

On The Of Decay Processes MHD Turbulence

Manish Kumar Singh¹, Raghvendra Mishra²

¹Department of mathematics NGBU Allahabad

²Department of mathematics Govt. PG College Ranikhet

Abstract: In this paper we have obtained the decay law of magnetic energy for the concentration fluctuation before the final period in rotating frame in the presence of dust particles for the case of multi-point and multi-time by integrating the energy spectrum over all wave number.

Key words: MHD turbulence, dusty turbulence, rotation, energy spectrum, first order reaction.

I. Introduction

Deissler (1958,1960) developed a theory “decay of homogenous turbulence for times before the final period” Using Deissler theory many other authors as Kishor and Sarkar (1991), Sarkar et.al (2003), Azad (2010), Rahaman (2010), Sukla (2011), Dixit (2010) discussed the decay process of MHD turbulence. Following Dixit (2010) we have extended the case of multipoint and single time for the case of multipoint and multi-time in a rotating system in presence of dust particles. Here, it is assumed that back reaction of the magnetic field on the velocity field may be neglected. Thus we have decoupled the magnetic effect from the Navier-Stokes equation so that the velocity and velocity spectrum will be independent quantities in the whole analysis.

II. BASIC EQUATIONS

The equations of motion and continuity for viscous, incompressible dusty fluid MHD turbulent flow in a rotating system are given by:

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_k} (u_i u_k) = -\frac{\partial w}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_k \partial x_k} - 2 \epsilon_{mkl} \Omega_m u_i + f(u_i - v_i) \quad (2.1)$$

$$\frac{\partial h_i}{\partial t} + \frac{\partial}{\partial x_k} (h_i u_k - u_i h_k) = \lambda \frac{\partial^2 h_i}{\partial x_k \partial x_k} \quad (2.1)$$

$$\frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k} = -\frac{K}{m_s} (v_i - u_i) \quad (2.3)$$

with

$$\frac{\partial u_i}{\partial x_i} = \frac{\partial v_i}{\partial x_k} = \frac{\partial h_i}{\partial x_i} = 0 \quad (2.4)$$

III. TWO-POINT, TWO-TIME CORRELATION AND SPECTRAL EQUATIONS

Under the condition that (i) the turbulence and the concentration magnetic field are homogeneous (ii) the chemical reaction has no effect on the velocity field, and (iii) the reaction rate and the magnetic diffusivity are constant, the induction equation of a magnetic field fluctuation of concentration of a dilute contaminant undergoing a first order chemical reaction at the points p p' and separated by the vector $\hat{\gamma}$ could be written as:

$$\frac{\partial h_i}{\partial t} + u_k \frac{\partial h_i}{\partial x_k} - h_k \frac{\partial u_i}{\partial x_k} = \lambda \frac{\partial^2 h_i}{\partial x_k \partial x_k} - R h_i \quad (3.1)$$

and

$$\frac{\partial h'_j}{\partial t'} + u'_k \frac{\partial h'_j}{\partial x'_k} - h'_k \frac{\partial u'_j}{\partial x'_k} = \lambda \frac{\partial^2 h'_j}{\partial x'_k \partial x'_k} - R h'_j \quad (3.2)$$

where, R is the constant reaction rate.

Multiplying Eq. (3.1) by h'_j and Eq. (3.2) by h_i and taking ensemble average, we get

$$\frac{\partial \langle h_i h'_j \rangle}{\partial t} + \frac{\partial}{\partial x_k} [\langle u_k h_i h'_j \rangle - \langle u_i h_k h'_j \rangle] = \lambda \frac{\partial^2 \langle h_i h'_j \rangle}{\partial x_k \partial x_k} - R \langle h_i h'_j \rangle \quad (3.3)$$

and

$$\frac{\partial \langle h_i h'_j \rangle}{\partial t'} + \frac{\partial}{\partial x'_k} [\langle u'_k h_i h'_j \rangle - \langle u'_j h_i h'_k \rangle] = \lambda \frac{\partial^2 \langle h_i h'_j \rangle}{\partial x'_k \partial x'_k} - R \langle h_i h'_j \rangle \quad (3.4)$$

Angular bracket $\langle \dots \rangle$ is used to denote an ensemble average.

Using the transformations

$$\frac{\partial}{\partial x_k} = -\frac{\partial}{\partial r_k}, \quad \frac{\partial}{\partial x'_k} = \frac{\partial}{\partial r_k}$$

$$\left(\frac{\partial}{\partial t}\right)_{t'} = \left(\frac{\partial}{\partial t}\right)_{\Delta t} - \frac{\partial}{\partial \Delta t}, \quad \frac{\partial}{\partial t'} = \frac{\partial}{\partial \Delta t} \quad (3.5)$$

Into Eq. (3.3) and (3.4), and writing in spectral form after contraction of indices as i and j we have.

$$\frac{\partial \langle \psi_i \psi_i \rangle}{\partial t} + 2[\lambda k^2 + R] \langle \psi_i \psi_i \rangle = 2iK_k \left[\langle \alpha_i \psi_k \psi_i \rangle (K, \Delta t, t) \right] - \left[\langle \alpha_k \psi_i \psi_i \rangle (-K, -\Delta t, t + \Delta t) \right] \quad (3.15)$$

and

$$\frac{\partial \langle \psi_i \psi_i \rangle}{\partial \Delta t} + [\lambda k^2 + R] \langle \psi_i \psi_i \rangle = iK_k \left[\langle \alpha_i \psi_k \psi_i \rangle (K, \Delta t, t) \right] - \left[\langle \alpha_k \psi_i \psi_i \rangle - \left(\tilde{K}, -\Delta t, t + \Delta t \right) \right] \quad (3.16)$$

The terms on the right side of Eq. (3.15) and (3.16) are collectively proportional to what is known as the magnetic energy transfer terms.

IV. THREE-POINT, THREE-TIME CORRELATION AND SPECTRAL EQUATIONS

Similar procedure can be used to find the three-point correlation equations. For this purpose we take the momentum equation of dusty fluid MHD turbulence in a rotating system at the point P and the induction equations of magnetic field fluctuations, governing the concentration of a dilute contaminant undergoing a first order chemical reaction at p' and p'' separated by the vector $\hat{\gamma}$ and $\hat{\gamma}'$ as:

$$\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} = -\frac{\partial \omega}{\partial x_i} + \frac{\partial^2 u_i}{\partial x_k \partial x_k} - 2 \epsilon_{mkl} \Omega_m u_l + f(u_i - v_i) \quad (4.1)$$

$$\frac{\partial h_i'}{\partial t} + u_k' \frac{\partial h_j'}{\partial x_k} - h_k' \frac{\partial u_i'}{\partial x_k} = \lambda \frac{\partial h_i'}{\partial x_k \partial x_k} - R h_i' \quad (4.2)$$

$$\frac{\partial h_i''}{\partial t} + u_k'' \frac{\partial h_j''}{\partial x_k} - h_k'' \frac{\partial u_i''}{\partial x_k} = \lambda \frac{\partial h_i''}{\partial x_k \partial x_k} - R h_i'' \quad (4.3)$$

Multiplying Eq. (4.1) by $h_i' h_j''$, Eq. (4.2) by $u_i' h_j''$ and Eq. (4.3) by $u_i' h_j''$, taking ensemble average, and contracting indices i and j we can write in spectral form as

$$\begin{aligned} & \frac{\partial}{\partial \Delta t} \langle \phi_i \beta_i' \beta_j'' \rangle \left(\hat{K}, \hat{K}', \Delta t, \Delta t', t \right) + \lambda \left[K^2 + \frac{R}{\lambda} \right] \langle \phi_i \beta_i' \beta_i'' \rangle \left(\hat{K}, \hat{K}', \Delta t, \Delta t', t \right) \\ & = -i k_k \langle \phi_i \phi_k' \beta_i' \beta_i'' \rangle \left(\hat{K}, \hat{K}', \Delta t, \Delta t', t \right) + i k_k \langle \phi_i \phi_k' \beta_i' \beta_i'' \rangle \left(\hat{K}, \hat{K}', \Delta t, \Delta t', t \right) \end{aligned} \quad (4.23)$$

And

$$\begin{aligned} & \frac{\partial}{\partial \Delta t'} k_i \langle \phi_i \beta_i' \beta_j'' \rangle \left(\hat{K}, \hat{K}', \Delta t, \Delta t', t \right) + \lambda \left[K'^2 + \frac{R}{\lambda} \right] \langle \phi_i \beta_i' \beta_i'' \rangle \left(\hat{K}, \hat{K}', \Delta t, \Delta t', t \right) \\ & = -i k_k' \langle \phi_i \phi_k' \beta_i' \beta_i'' \rangle \left(\hat{K}, \hat{K}', \Delta t, \Delta t', t \right) + i k_k' \langle \phi_i \phi_k' \beta_i' \beta_i'' \rangle \left(\hat{K}, \hat{K}', \Delta t, \Delta t', t \right) \end{aligned} \quad (4.24)$$

V. SOLUTIONS FOR TIMES BEFORE THE FINAL PERIOD

It is well known that the equation for final period of decay is obtained by considering the two-point correlations after neglecting third-order correlation terms. To study the decay for times before the final period, the three-point correlations are considered and the quadruple correlation terms are neglected because the quadruple correlation terms decays faster than the lower-order correlation terms. The term $\langle \gamma \beta_i' \beta_j'' \rangle$ associated with the pressure fluctuations should also be neglected. Thus neglecting all the terms on the right hand side of Eq. (4.4) to (4.6). On integrating between t_0 and t and letting $r = 0$ and following Dixit (2010) we get,

$$\frac{\partial E}{\partial t} + 2\lambda k^2 E = F \quad (5.1)$$

Where, $E = 2\pi k^2 \langle \psi_i \psi_i' \rangle$, E is the magnetic energy spectrum function and F is the magnetic energy transfer term and is given by:

$$\begin{aligned} F &= -2\delta \int_0^\infty (k^2 k'^4 - k^4 k'^2) k^2 k'^2 \\ & \times \left[\int_{-1}^1 \exp \left\{ -\lambda [1 + P_M] (k^2 + k'^2) (t - t_0) \right. \right. \\ & \left. \left. + k^2 \Delta t + 2P_M (t - t_0) k k' \cos \theta \right. \right. \\ & \left. \left. + \frac{2R}{\lambda} \left(t - t_0 + \frac{\Delta t}{2} \right) \right. \right. \\ & \left. \left. + \left(\frac{2 \epsilon_{mkl} \Omega_m}{\lambda} - \frac{fs}{\lambda} \right) \right] d(\cos \theta) dk' \end{aligned} \quad (5.1)$$

Integrating Eq. (5.13) with respect to $\cos \theta$ and we have:

$$\begin{aligned}
 F = & -\frac{\delta_0 P_M \sqrt{\pi}}{4\lambda^{3/2} (t-t_0)^{3/2} (1+P_M)^{5/2}} \\
 & \exp \left\{ -\left(\frac{2\epsilon_{mkl} \Omega_m}{\lambda} - \frac{fs}{\lambda} \right) (t-t_0) \right\} \times \\
 & \exp \left[\frac{-k^2 \lambda (1+2P_M)}{1+2P_M} \left(t-t_0 + \frac{1+P_M}{1+2P_M} \Delta t \right) \right. \\
 & \left. -2R \left(t-t_0 + \frac{\Delta t}{2} \right) \right] \times \left[\frac{15P_M k^4}{4P_M^2 \lambda^2 (t-t_0)^2 (1+P_M)} \right. \\
 & \left. + \left\{ \frac{5P_M^2}{(1+P_M)^2} - \frac{3}{2} \right\} \frac{k^6}{P_M \lambda (t-t_0)} \right. \\
 & \left. + \left\{ \frac{P_M^3}{(1+P_M)^2} - \frac{P_M}{1+P_M} \right\} k^8 \right] \\
 & -\frac{\delta_0 P_M \sqrt{\pi}}{4\lambda^{3/2} (t-t_0 + \Delta t)^{3/2} (1+P_M)^{5/2}} \\
 & \exp \left\{ -\left(\frac{2\epsilon_{mkl} \Omega_m}{\lambda} - \frac{fs}{\lambda} \right) (t-t_0) \right\} \times \\
 & \exp \left[\frac{-k^2 \lambda (1+2P_M)}{1+2P_M} \left(t-t_0 + \frac{P_M}{1+P_M} \Delta t \right) \right. \\
 & \left. -2R \left(t-t_0 + \frac{\Delta t}{2} \right) \right] \times \left[\frac{15P_M k^4}{4v^2 (t-t_0 + \Delta t)^2 (1+P_M)} \right. \\
 & \left. + \left\{ \frac{5P_M^2}{(1+P_M)^2} - \frac{3}{2} \right\} \frac{k^6}{P_M \lambda (t-t_0 + \Delta t)} \right. \\
 & \left. + \left\{ \frac{P_M^2}{(1+P_M)^3} - \frac{P_M}{(1+P_M)} \right\} k^8 \right] \tag{5.2}
 \end{aligned}$$

The series of Eq. (5.2) contains only even power of k and start with k⁴ and the equation represents the transfer function arising owing to consideration of magnetic field at three-point and three-times.

If we integrate Eq. (5.2) for $\square t = 0$ over all wave numbers, we find that:

$$\int_0^\infty F dk = 0 \tag{5.3}$$

which indicates that the expression for F satisfies the condition of continuity and homogeneity. Physically it was to be expected as F is a measure of the energy transfer and the total energy transferred to all wave numbers must be zero.

The linear Eq. (5) can be solved to give:

$$E = \exp \left[-2\lambda k^2 \left(t - t_0 + \frac{\Delta t}{2} \right) \right] \int_0^\infty \exp \left[2\lambda \left(k^2 + \frac{R}{\lambda} \right) \left(t - t_0 + \frac{\Delta t}{2} \right) \right] dt + J(k) \exp \left[-2\lambda \left(k^2 + R \left(t - t_0 + \Delta t \right) \right) \right] \quad (5.4)$$

where $J(k) = \frac{N_0 k^2}{\pi}$ is a constant of integration and can be obtained as by Corrsin (1951). Substituting the values of F from Eq. (5.2) into (5.4) gives the equation:

$$E = \frac{N_0 k^2}{\pi} \exp \left[-2\lambda \left(k^2 + \frac{R}{\lambda} \right) \left(t - t_0 + \frac{\Delta t}{2} \right) \right] + \frac{\delta_0 P_M \sqrt{\pi}}{4\lambda^{3/2} (1 + P_M)^{1/2}} \times \exp \left[-(2 \epsilon_{mkl} \Omega_m - fs)(t - t_0) \right] \exp \left[\frac{-k^2 \lambda (1 + 2P_M)}{1 + P_M} \left(t - t_0 + \frac{1 + P_M}{1 + 2P_M} \Delta t \right) - t - t_0 + \frac{\Delta t}{2} \right] \left[\frac{3k^4}{2P_M \lambda^2 (t - t_0)^{5/2}} + \frac{(7P_M - 6)k^6}{3\lambda (1 + P_M) (t - t_0)^{3/2}} - \frac{4(3P_M^2 - 2P_M + 3)k^8}{3(1 + P_M)^2 (t - t_0)^{1/2}} + \frac{8\sqrt{\lambda} (3P_M^2 - 2P_M + 3)k^9 F(\omega)}{(1 + P_M)^{5/2} F_M^{1/2}} \right] \quad (5.5)$$

where, $F(\omega) = e^{-\omega^2} \int_0^\infty e^{x^2} dx$

$$\omega = k \sqrt{\frac{\lambda (t - t_0)}{1 + P_M}} \text{ or } k \sqrt{\frac{\lambda (t - t_0 + \Delta t)}{1 + P_M}}$$

By setting $r^\wedge = 0, j=i, dk = -2\pi k^2 d(\cos \theta) d\hat{k}$ and $E = 2\pi k^2 \langle \psi_i \psi_j \rangle$ in Eq. (5.2) we get the expression for magnetic energy decay law as:

$$\frac{\langle h_i h_i \rangle}{2} = \int_0^\infty E dk \quad (5.6)$$

Substituting Eq. (5.5) into (5.6) and after integration, we get:

$$\begin{aligned}
 \frac{\langle h_i h_i' \rangle}{2} &= \frac{N_0}{8\sqrt{2\pi} \lambda^{3/2} \left(T + \frac{\Delta T}{2}\right)^{3/2}} \\
 &\quad \exp \left[-2R \left(T + \frac{\Delta T}{2}\right) \right] \\
 &+ \frac{\pi \delta_0}{4\lambda^6 (1+P_M)(1+2P_M)^{5/2}} \\
 &\quad \exp \left[-2R \left(T + \frac{\Delta T}{2}\right) \right] \exp \left[-(2\epsilon_{mkl} \Omega_m - fs) \right] \\
 &\times \left[\frac{9}{16T^{5/2} \left(T + \frac{1+P_M}{1+2P_M} \Delta T\right)^{3/2}} \right. \\
 &\quad + \frac{9}{16(T + \Delta T)^{5/2} \left(T + \frac{P_M}{1+2P_M} \Delta T\right)^{5/2}} \\
 &\quad + \frac{5P_M(7P_M - 6)}{16(1+2P_M)T^{3/2} \left(T + \frac{P_M}{1+2P_M} \Delta T\right)^{7/2}} \\
 &\quad + \frac{5P_M(7P_M - 6)}{16(1+2P_M)(T + \Delta T)^{3/2} \left(T + \frac{P_M}{1+2P_M} \Delta T\right)^{7/2}} \\
 &\quad + \frac{35P_M(3P_M^2 - 2P_M + 3)}{8(1+2P_M)T^{1/2} \left(T + \frac{1+P_M}{1+2P_M} \Delta T\right)^{9/2}} \\
 &\quad + \frac{35P_M(3P_M^2 - 2P_M + 3)}{8(1+2P_M)(T + \Delta T)^{1/2} \left(T + \frac{P_M}{1+2P_M} \Delta T\right)^{9/2}} \\
 &\quad + \frac{8P_M(3P_M^2 - 2P_M + 3)(1+2P_M)^{5/2}}{3.2^{23/2} (1+2P_M)^{11/2}} \\
 &\quad \left. \sum_{n=0}^{\infty} \frac{1.3.4 \dots (2n+9)}{n(2n+1)2^{2n} (1+P_M)^n} \times \right.
 \end{aligned}$$

$$\left\{ \frac{T^{(2n+1)/2}}{\left(T + \frac{\Delta T}{2}\right)^{(2n+1)/2}} + \frac{(T + \Delta T)^{(2n+1)/2}}{\left(T + \frac{\Delta T}{2}\right)^{(2n+1)/2}} \right\} \quad (5.7)$$

Where $T = t - t_0$.

For $T_m = T + \frac{\Delta T}{2}$, Eq. 5.19 takes the form:

$$\begin{aligned} \frac{\langle h^2 \rangle}{2} &= \frac{\langle h_i h_i \rangle}{2} = \exp[-2RT_M] \\ &\left[\frac{N_0}{8\sqrt{2\pi} \lambda^{3/2} T_m^{3/2}} + \frac{\pi \delta_0}{4\lambda^6 (1+P_M)(1+2P_M)^{5/2}} \right. \\ &\exp[-(2\epsilon_{mkl} \Omega_m - fs)] \\ &\times \left[\frac{9}{16\left(T_M - \frac{\Delta T}{2}\right)^{5/2} \left(T_M + \frac{\Delta T}{1+2P_M}\right)^{5/2}} \right. \\ &+ \frac{9}{16\left(T_M - \frac{\Delta T}{2}\right)^{5/2} \left(T_M + \frac{\Delta T}{2(1+2P_M)}\right)^{5/2}} \\ &+ \frac{5P_M(2P_M - 6)}{16(1+2P_M)\left(T_M - \frac{\Delta T}{2}\right)^{3/2} \left(T_M + \frac{\Delta T}{2(1+2P_M)}\right)^{7/2}} \\ &+ \left. \frac{5P_M(2P_M - 6)}{16(1+2P_M)\left(T_M + \frac{\Delta T}{2}\right)^{3/2} \left(T_M - \frac{\Delta T}{2(1+2P_M)}\right)^{7/2}} + \dots \right] \end{aligned} \quad (5.8)$$

This is the decay law of magnetic energy fluctuations of concentration of a dilute contaminant undergoing a first order chemical reaction before the final period for the case of multi-point and multi-time in MHD turbulence in a rotating system in presence of dust particle.

VI. RESULTS AND DISCUSSION

In Eq. (5.6) we obtained the decay law of magnetic energy fluctuations of a dilute contaminant undergoing a first order chemical reaction before the final period considering three-point correlation terms for the case of multi-point and multi-time in MHD turbulence in presence of dust particle in a rotating system.

If the fluid is non-rotating and clean then, $f = 0$, the Eq. (5.6) becomes:

$$\begin{aligned}
 \frac{\langle h^2 \rangle}{2} &= \exp[-2RT_M] \\
 &\left[\frac{N_0}{8\sqrt{2\pi} \lambda^{3/2} T_M^{3/2}} + \frac{\pi\delta_0}{4\lambda^6 (1+P_M)(1+2P_M)^{5/2}} \right. \\
 &\times \left[\frac{9}{16 \left(T_M - \frac{\Delta T}{2}\right)^{5/2} \left(T_m + \frac{\Delta T}{1+2P_M}\right)^{5/2}} \right. \\
 &+ \left. \frac{9}{16 \left(T_M - \frac{\Delta T}{2}\right)^{5/2} \left(T_m + \frac{\Delta T}{1+2P_M}\right)^{5/2}} \right. \\
 &\left. \frac{5P_M(7P_M-6)}{16(1+2P_M) \left(T_M - \frac{\Delta T}{2}\right)^{3/2} \left(T_m + \frac{\Delta T}{2(1+2P_M)}\right)^{7/2}} \right. \\
 &+ \left. \frac{5P_M(7P_M-6)}{16(1+2P_M) \left(T_M + \frac{\Delta T}{2}\right)^{3/2} \left(T_m - \frac{\Delta T}{2(1+2P_M)}\right)^{7/2}} \right] + \dots
 \end{aligned} \tag{6.1}$$

Which was obtained earlier by Islam and Sarker (2001).

If we Put $\Delta T = 0, R = 0$, in Eq. 6.1 we can easily find out.

$$\begin{aligned}
 \frac{\langle h^2 \rangle}{2} &= \frac{\langle h_i h_i \rangle}{2} = \frac{N_0 T^{-3/2}}{8\sqrt{2\pi} \lambda^{3/2}} \\
 &+ \frac{\pi\delta_0}{4 \lambda^6 (1+P_M)(1+2P_M)^{5/2}} T^{-5} \\
 &\left\{ \frac{9}{16} + \frac{5 P_M (7P_M - 6)}{16 (1 + 2P_M)} + \dots \right\}
 \end{aligned} \tag{6.2}$$

Which is same as obtained earlier by Sarker and Kishore (1991)?

This study shows that due to the effect of rotation of fluid in the flow field with chemical reaction of the first order in the concentration the magnetic field fluctuation in dusty fluid MHD turbulence in rotating system for the case of multi-point and multi-time, i.e., the turbulent energy decays more rapidly than the energy for non-rotating clean fluid and the faster rate is governed by $\exp[-(2 \epsilon_{mkl} \Omega_m - fs)]$. Here the chemical reaction in MHD turbulence for the case of multi-point And, multi-time, causes.

The concentration to decay more they would for non-rotating clean fluid and it is governed by $\exp[-(2RT_m + 2 \epsilon_{mkl} \Omega_m - fs)]$. The first term of right hand side of Eq. (5.8) corresponds to the energy of magnetic field fluctuation of concentration for the two-point correlation and the second term represents magnetic energy for the three-point correlation. In Eq. (5.6), the term associated with the three-point correlation dies out faster than the two-point correlation. If higher order correlations are considered in the analysis, it appears that more terms of higher power of time would be added to the Eq. (5.6). For larger times the last term in the Eq. (5.6) becomes negligible, leaving the -3/2 power decay law for the final period.

REFERENCE

1. Azad M.A. (2010) Res Journal of maths and stat 2(2),56
2. Deissler R.G (1960) Phys. Fluid3,176
3. Dixit T. (2010) ARJPS 13 (1,2),61
4. Kishor N. and sarkar MSA (1991) Int.J Eng. Sc.29,14,79
5. Rahaman MI (2010) J. Mech. Cont. Math, Sc 4,509
6. Sukla A (2011) ARJPS 14(1) 21

