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Mathematical Modelling of Ecosystem Sustainability in the Presence of Nutrients, Consumers and Predator Food Web

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Abstract—This paper examines a mathematical modelling of ecosystem sustainability in the presence of nutrients, consumers and predator food web. A nonlinear mathematical model for nutrient, consumer and predator including the dependence of predator on both nutrients and consumers is proposed and analysed using a system of differential equations. The boundedness and conditions for existence and stability of the local equilibrium points of the system are analysed qualitatively. The results show that the system has stable as well as unstable equilibrium points. Numerical simulations of the model are carried out to investigating the effects of predator dependence on both nutrients and consumers. The model shows that through the predator dependence on both nutrients and consumers, predator population increases.

Keywords—Ecosystem; Nutrient; Consumer; Predator; Food web.

I. INTRODUCTION

Ecosystems are made up of abiotic (non-living, environmental) and biotic components and these basic components are important to nearly all types of ecosystem [1]. In nature, fluxes across habitat often bring both nutrient and energetic resources into areas of low productivity from areas of higher productivity. These inputs can alter consumption rates of consumer and predator species in the recipient food webs, thereby influencing food web stability of a simple food web model [2]. Food web models can predict that nutrient enrichment can decrease food web stability as it can amplify variability in predator-prey cycles and even extirpate predator populations. Mathematical models developed in the theoretical ecology predict complex food webs and are less stable than simple webs [3]. Consumers are typically viewed as predatory animals such as the wolf and hyena. Consumers have an important role to play within an ecosystem such as balancing the food chain by keeping animal populations at a reasonable number. Without proper balance, an ecosystem can collapse and cause the decline of all affected species. This will lead to a severe affected ecosystem, and non-functional food web [4]. Beyond the direct impacts of predator-prey interactions, trophic cascades can be either weakened or intensified by behavioural responses of both prey and predator. Plants interact through the exploitative competition mediated by a predator. Community stability is ensured if plants have trade-off between their abilities to exploit resources and to defend against consumers [5;6]. Ecologists study these interactions in order to understand the abundance and diversity of life within earth’s ecosystem [7].

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II. MODEL FORMULATION

A nonlinear mathematical model for nutrient, consumer, predator interaction and the dependence of predator on both nutrients and consumers is proposed and analysed. The proposed model is divided into three classes namely $X(t)$ which denotes the amount of nutrients present in the system, $Y(t)$ the number of consumer species and $Z(t)$ the number of predator species at time $t > 0$.

In formulating the model the following assumptions are taken into account:

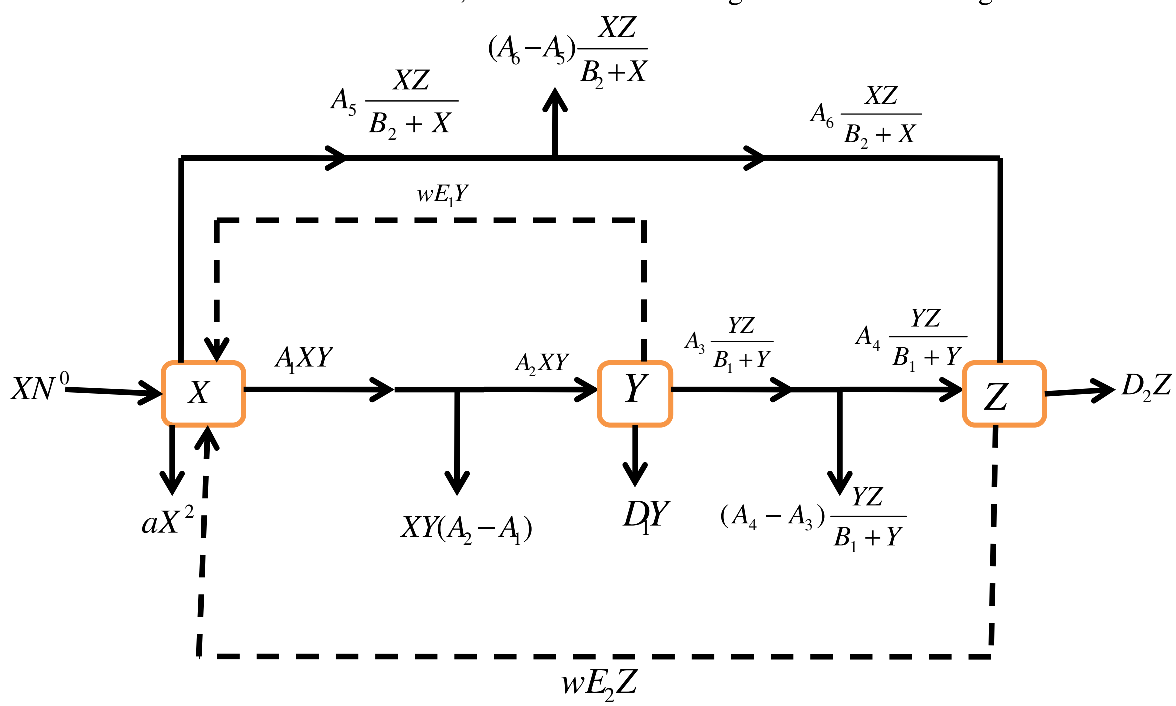
(i) Predator is provided with additional food of constant biomass A which is distributed uniformly in the habitat.

(ii) The number of encounters per predator with the additional food is proportional to the density of the additional food.

(iii) The proportionality constant characterizes the ability of the predator to identify the additional food.

(iv) The predator depends on both nutrient and consumers at a constant conversion rates A_5 and A_6. The half-saturation constants for predator are B_1 and B_2.

Taking into account the above considerations, we have the following schematic flow diagram:



Figuere 1: Model flow chart for nutrient, consumer and predator and the dependence of predator on both nutrients and consumers

From figure (1) the model flow chart is governed by the following system of nonlinear ordinary differential equations:

$$\frac{dX}{dT} = X\left(N^0 - aX\right) - A_1XY + w\left(E_1Y + E_2Z\right) - A_5\frac{XZ}{B_2+X},$$

$$\frac{dY}{dT} = A_2XY - A_3\frac{YZ}{B_1+Y} - D_1Y, \qquad (1)$$

$$\frac{dZ}{dT}=A_4\frac{YZ}{B_1+Y}+A_6\frac{XZ}{B_2+X}-D_2Z.$$

With the initial conditions $X(0)>0$, $Y(0)>0$ and $Z(0)>0$.

where N^0 is the constant rate of nutrients supply in the system; A_1 and A_2 are conversion rates of nutrients supply to consumers; A_3 and A_4 are conversion rates of consumer to predator for species Y and Z; A_5 and A_6 are conversion rates of nutrients supply to predators; D_1 and D_2 are death rates of consumers and predators; wE_1 and wE_2 are nutrient regeneration from dead consumer and predator population; T is time; B_1 and B_2 are half saturation constants for predator population; a is leaching rate. In order to reduce the number of parameters and to determine which combinations of parameters control the behaviour of the system (1), the parameters and variables are renamed as $N=X$, $C=Y/B_1$, $P=Z$, $t=T$, $d_1=D_1$, $d_2=D_2$, $\alpha_1=A_1B_1$, $\alpha_2=A_2B_1$, $\gamma_1=E_1B_1$, $\gamma_2=E_2$, $w=\omega$, $\beta=A_3$, $\beta_1=A_4$, $\beta_2=A_5$, $\beta_3=A_6$; to obtain the following equations:

$$\begin{aligned}\frac{dN}{dt}&=N\left(N^0-aN\right)-\alpha_1NC+\omega\left(\gamma_1C+\gamma_2P\right)-\frac{\beta_2NP}{1+N},\\ \frac{dC}{dt}&=\alpha_2NC-\frac{\beta CP}{1+C}-d_1C,\\ \frac{dP}{dt}&=\frac{\beta_1CP}{1+C}+\frac{\beta_3NP}{1+N}-d_2P.\end{aligned} \tag{2}$$

With the initial conditions: $N(0)>0$, $C(0)>0$, and $P(0)>0$.

III. MODEL ANALYSIS

Model (2) is analysed qualitatively to get insights into its dynamical features which is envisaged to give better understanding of the effects of increase of nutrients and consumers on the growth rate of predator population. Since the state variables N, C and P represent populations, it is assumed that they are positive for $t\geq 0$.

3.1 BOUNDEDNESS OF THE MODEL

Theorem 1

All the solutions of the system (2) which initiate at $\mathbb{R}^3_+$ are uniformly bounded [8].

Proof: Let $\left(N(t),\ C(t),\ P(t)\right)$ be any solution of the system (2).

Let $\varepsilon=N+C+P$.

Then we have

$$\frac{d\varepsilon}{dt}=\frac{dN}{dt}+\frac{dC}{dt}+\frac{dP}{dt} \tag{3}$$

Substituting the system (2) into (3) and simplifying, we get

$$\frac{d\varepsilon}{dt}=N\left(N^0-aN\right)-\alpha_1NC+\omega\left(\gamma_1C+\gamma_2P\right)$$
$$-\frac{\beta_2NP}{1+N}+\alpha_2NC-\frac{\beta CP}{1+C}-d_1C+\frac{\beta_1CP}{1+C}+\frac{\beta_3NP}{1+N}-d_2P. \tag{4}$$

or

$$\frac{d\varepsilon}{dt} = NN^0 - aN^2 - NC(\alpha_1 - \alpha_2) + \omega(\gamma_1 C + \gamma_2 P) - \frac{CP(\beta - \beta_1)}{1+C} - \frac{NP(\beta_2 - \beta_3)}{1+N} - d_1 C - d_2 P. \quad (5)$$

Assuming that, $\alpha_1 = \alpha_2$, $\beta = \beta_1$, and $\beta_2 = \beta_3$, where α_1, α_2, β, β_1, β_2 and β_3 are conversion rates, it follows that

$$\frac{d\varepsilon}{dt} \le \left(2NN^0 + \omega\gamma_1 C + \omega\gamma_2 P\right) - K\left(N + C + P\right) \quad (6)$$

where $K = \min(N^0, d_1, d_2)$.

Since

$$\frac{d\varepsilon}{dt} \le (2NN^0 + \omega\gamma_1 C + \omega\gamma_2 P) - K\varepsilon$$

then

$$\frac{d\varepsilon}{dt} + K\varepsilon \le \theta\left(2N^0 + \omega\gamma_1 + \omega\gamma_2\right) \quad (7)$$

where $\theta = \max\left\{N(0), N^0/a, C(0), P(0)\right\}$.

The inequality (7) has a solution

$$\varepsilon \le \frac{\theta\left(2N^0 + \omega\gamma_1 + \omega\gamma_2\right)}{K}\left(1 - e^{-Kt}\right) + \varepsilon e^{-Kt}. \quad (8)$$

As $t \to \infty$, then $\varepsilon \le \dfrac{\theta\left(2N^0 + \omega\gamma_1 + \omega\gamma_2\right)}{K}$.

Hence, all the solutions of the system (2) that initiated in $\mathbb{R}^3_+$ are confined in the region

$$B = \left\{(N, C, P) \in \mathbb{R}^3_+ : 0 < \varepsilon \le \frac{\theta\left(2N^0 + \omega\gamma_1 + \omega\gamma_2\right)}{K}\right\} \quad (9)$$

3.2 POSITIVITY OF THE SOLUTIONS

Theorem 2

Let $\{N(0) > 0,\ C(0) > 0,\ P(0) > 0\} \in B$, where B is the invariant region. Then the solution of the system (2) is positive for $\forall t \ge 0$.

Proof:

To prove the theorem we use the equations of the system (2). From the first equation of the system (2), we have

$$\frac{dN}{dt} = N\left(N^0 - aN\right) - \alpha_1 NC + \omega\left(\gamma_1 C + \gamma_2 P\right) - \frac{\beta_2 NP}{1+N} \quad (10)$$

From the equation (10) we obtain

$$\frac{dN}{dt} \le N\left(N^0 - aN\right) \quad (11)$$

which gives

$$N(t) \le \frac{N^0 H}{e^{-N^0 t} + aH} \quad (12)$$

At $t=0$ it yields

$$N(0) \le \frac{N^0 H}{1+aH} \text{ and } H \ge \frac{N(0)}{N^0 - aN(0)} \tag{13}$$

Substitute equation (13) into equation (12) to obtain

$$N(t) \le \frac{N^0 N(0)}{e^{-N^0 t}\left(N^0 - aN(0)\right) + aN(0)}. \tag{14}$$

As $t \to \infty$ we obtain $0 < N(t) \le \frac{N^0}{a}$, $\forall t \ge 0.$

Similarly using the second and third equation of system (2) positivity of solutions can be established. Hence, all the solutions of the system (2) that are initiated in $\mathbb{R}_+^3$ are confined in the region $B = \left\{(N, C, P) \in \mathbb{R}_+^3\right\}$.

3.3 STABILITY ANALYSIS OF EQUILIBRIA

The equilibrium points of system (2) are obtained by setting the time derivative of system (2) equal to zero. That is $N'(t) = C'(t) = P'(t) = 0.$

The model system (2) consists of four equilibrium points.

(i)The trivial equilibrium is $E_0(0,0,0)$

(ii) The axial equilibrium point is $E_1\left(\frac{N^0}{a}, 0, 0\right)$

(iii) The predator-free equilibrium point is $E_2\left(N^*, C^*, 0\right)$, where

$$N^* = \frac{d_1}{\alpha_2} \tag{15}$$

$$C^* = -\frac{d_1(ad_1 - b\alpha_2)}{\alpha_2(d_1\alpha_1 - \omega\alpha_2\gamma_1)} \tag{16}$$

(v) The existence and local stability of interior equilibrium point $E^*\left(N^*, C^*, P^*\right)$.

The interior equilibrium point of the system (2) is given by $E^*\left(N^*, C^*, P^*\right)$, where

$C^* = \frac{d_2(1+N^*) - \beta_3 N^*}{\beta_1(1+N^*) + \beta_3 N^* - d_2(1+N^*)}$, $P^* = \frac{(\alpha_2 N^* - d_1)(1+C^*)}{\beta}$ and N^* is obtained as the positive roots of the equation

$$AN^{*3} + BN^{*2} + DN^* + E = 0.$$

where

$$A = \beta^2 d_2 a\beta_3 - a\beta^2\beta_1\beta_3,$$

$$B = a\beta\beta_3 + \beta^2\beta_3 d_2 a - \beta^2\beta_3 d_2 N^0 - \beta^2\beta_3 d_2\alpha_1 - a\beta_1\beta^2\beta_3 + N^0\beta_1\beta^2\beta_3 - \beta\beta_2\beta_3\beta_1\alpha_2 + \alpha_2\gamma_2\omega\beta\beta_1\beta_3,$$

$$D = N^0\beta\beta_3 + \beta\beta_3\alpha_1 - \beta^2\beta_3 d_2 N^0 - \beta^2\beta_3 d_2\alpha_1 + \beta^2\beta_3 d_2\omega\gamma_1 + \beta^2\beta_3 N^0\beta_1 + \beta\beta_1\beta_2\beta_3 d_1 - \omega\gamma_2\beta\beta_1\beta_3 d_1 + \alpha_2\omega\gamma_2\beta\beta_1\beta_3,$$

$E = \beta\beta_3\omega\gamma_1 + \beta^2\beta_3 d_2\omega\gamma_1 - \omega\gamma_2\beta\beta_1\beta_3 d_1.$

Therefore, the interior equilibrium point E^* exists if

$$d_2\left(1+N^*\right) > \beta_3 N^*,\ \beta_1\left(1+N^*\right)+\beta_3 N^* > d_2\left(1+N^*\right),\ \alpha_2 N^* > d_1,\ \beta \neq 0 \text{ and}$$
$$AN^{*3} + BN^{*2} + DN^* + E \geq 0.$$

The local stability of the equilibrium points is obtained by using the Jacobian matrix of the system (2), that is:

$$\mathbf{J}_{(N,C,P)} = \begin{pmatrix} \frac{\partial f}{\partial N} & \frac{\partial f}{\partial C} & \frac{\partial f}{\partial P} \\ \frac{\partial g}{\partial N} & \frac{\partial g}{\partial C} & \frac{\partial g}{\partial P} \\ \frac{\partial h}{\partial N} & \frac{\partial h}{\partial C} & \frac{\partial h}{\partial P} \end{pmatrix}_{(N,C,P)}.$$

This gives

$$\mathbf{J}_{(N,C,P)} = \begin{pmatrix} N^0 - 2aN^* - \alpha_1 C^* + \frac{\beta_2 N^* P^*}{\left(1+N^*\right)^2} - \frac{\beta_2 P^*}{\left(1+N^*\right)} & -\alpha_1 N^* + \omega\gamma_1 & \omega\gamma_2 - \frac{\beta_2 N^*}{1+N^*} \\ \alpha_2 C^* & \alpha_2 N^* + \frac{\beta C^* P^*}{\left(1+C^*\right)^2} - \frac{\beta P^*}{\left(1+C^*\right)} - d_1 & -\frac{\beta C^*}{1+C^*} \\ -\frac{\beta_3 N^* P^*}{\left(1+N^*\right)^2} + \frac{\beta_3 P^*}{\left(1+N^*\right)} & -\frac{\beta_1 C^* P^*}{\left(1+C^*\right)^2} + \frac{\beta_1 P^*}{\left(1+C^*\right)} & \frac{\beta_1 C^*}{1+C^*} + \frac{\beta_3 N^*}{1+N^*} - d_2 \end{pmatrix} \quad (17)$$

The stability of the trivial, axial and predator-free equilibrium points can be summarised by the following theorem.

Theorem 3

The trivial equilibrium point E_0 is locally asymptotically unstable. The axial equilibrium point E_1 is locally asymptotically stable whenever $\frac{N^0\alpha_2}{a} < d_1$, $\frac{N^0\beta_3}{a+N^0} < d_2$ and unstable otherwise. The predator-free equilibrium point E_2 is locally asymptotically stable if $d_1\left(\frac{\beta_3}{d_1+\alpha_2} + \frac{\left(ad_1 - N^0\alpha_2\right)\beta_1}{d_1\left(ad_1 - \left(N^0+\alpha_1\right)\alpha_2\right) + \omega\alpha_2^2\gamma_1}\right) < d_2$, $\lambda_2 < 0$, and $\lambda_3 < 0$ [8].

Proof:

After linearizing the system at trivial equilibrium point $E_0(0,0,0)$, we obtain

$$\mathbf{J}_{(E_0)} = \begin{pmatrix} N^0 & \omega\gamma_1 & \omega\gamma_2 \\ 0 & -d_1 & 0 \\ 0 & 0 & -d_2 \end{pmatrix}. \quad (18)$$

The corresponding eigenvalues are:

$$\begin{bmatrix} N^0 \\ -d_1 \\ -d_2 \end{bmatrix} \tag{19}$$

One of the eigenvalue is positive and two are negative. So, the trivial equilibrium point E_0 always exist but unstable.

The Jacobian matrix of the axial equilibrium point $E_1\left(\frac{N^0}{a},0,0\right)$ after linearizing yields

$$\mathbf{J}_{(E_1)} = \begin{pmatrix} -N^0 & \omega\gamma_1 - \frac{N^0\alpha_1}{a} & \omega\gamma_2 - \frac{N^0\beta_2}{a+N^0} \\ 0 & \frac{\alpha_2 N^0}{a} - d_1 & 0 \\ 0 & 0 & \frac{N^0\beta_3}{a+N^0} - d_2 \end{pmatrix} \tag{20}$$

The corresponding eigenvalues are

$$\begin{bmatrix} -N^0 \\ \frac{N^0\alpha_2}{a} - d_1 \\ \frac{N^0\beta_3}{a+N^0} - d_2 \end{bmatrix} \tag{21}$$

Hence the axial equilibrium point E_1 is locally asymptotically stable and unstable otherwise since $\frac{\alpha_2 N^0}{a} < d_1$ and $\frac{\beta_3 N^0}{a+N^0} < d_2$.

The local stability of the predator-free equilibrium point $E_2\left(\frac{d_1}{\alpha_2}, -\frac{d_1\left(ad_1 - N^0\alpha_2\right)}{\alpha_2\left(d_1\alpha_1 - \omega\alpha_2\gamma_1\right)}, 0\right)$ is obtained from the following Jacobian matrix

$$\mathbf{J}_{(E_2)} = \begin{pmatrix} \frac{ad_1^2\alpha_1 + \omega\alpha_2\left(-2ad_1 + N^0\alpha_2\right)\gamma_1}{\alpha_2\left(-d_1\alpha_1 + \omega\alpha_2\gamma_1\right)} & -\frac{d_1\alpha_1}{\alpha_2} + \omega\gamma_1 & -\frac{d_1\beta_2}{d_1+\alpha_2} + \omega\gamma_2 \\ \frac{d_1\left(-ad_1 + N^0\alpha_2\right)}{d_1\alpha_1 - \omega\alpha_2\gamma_1} & 0 & \frac{\beta d_1\left(-ad_1 + N^0\alpha_2\right)}{ad_1^2 - d_1\left(N^0+\alpha_1\right)\alpha_2 + \omega\alpha_2^2\gamma_1} \\ 0 & 0 & -d_2 + d_1\left(\frac{\beta_3}{d_1+\alpha_2} + \frac{\left(ad_1 - N^0\alpha_2\right)\beta_1}{d_1\left(ad_1 - \left(N^0+\alpha_1\right)\alpha_2\right) + \omega\alpha_2^2\gamma_1}\right) \end{pmatrix} \tag{22}$$

The corresponding eigenvalues are

$$\lambda_1 = -d_2 + d_1\left(\frac{\beta_3}{d_1+\alpha_2} + \frac{\left(ad_1 - N^0\alpha_2\right)\beta_1}{d_1\left(ad_1 - \left(N^0+\alpha_1\right)\alpha_2\right) + \omega\alpha_2^2\gamma_1}\right),$$

$$\lambda_2 = \frac{\left(\begin{array}{l} N^0\omega\alpha_2^2\gamma_1 + ad_1\left(d_1\alpha_1 - 2\omega\alpha_2\gamma_1\right) - \\ \sqrt{4d_1\alpha_2\left(ad_1 - N^0\alpha_2\right)\left(d_1\alpha_1 - \omega\alpha_2\gamma_1\right)^2 + \left(ad_1^2\alpha_1 + \omega\alpha_2\left(-2ad_1 + N^0\alpha_2\right)\gamma_1\right)^2} \end{array}\right)}{2\alpha_2\left(-d_1\alpha_1 + \omega\alpha_2\gamma_1\right)}$$

$$\lambda_3 = \frac{\left(\begin{array}{l} N^0\omega\alpha_2^2\gamma_1 + ad_1\left(d_1\alpha_1 - 2\omega\alpha_2\gamma_1\right) + \\ \sqrt{4d_1\alpha_2\left(ad_1 - N^0\alpha_2\right)\left(d_1\alpha_1 - \omega\alpha_2\gamma_1\right)^2 + \left(ad_1^2\alpha_1 + \omega\alpha_2\left(-2ad_1 + N^0\alpha_2\right)\gamma_1\right)^2} \end{array}\right)}{2\alpha_2\left(-d_1\alpha_1 + \omega\alpha_2\gamma_1\right)}$$

Therefore, the predator-free equilibrium point E_2 is asymptotically stable if $\lambda_2 < 0,\ \lambda_3 < 0$ and

$$d_1\left(\frac{\beta_3}{d_1+\alpha_2} + \frac{\left(ad_1 - N^0\alpha_2\right)\beta_1}{d_1\left(ad_1 - \left(N^0+\alpha_1\right)\alpha_2\right) + \omega\alpha_2^2\gamma_1}\right) < d_2. \quad (23)$$

Theorem 4

The interior equilibrium point $E^*\left(N^*, C^*, P^*\right)$ for the system (2) is locally asymptotically stable if the following conditions hold: $\Omega_1 > 0,\ \Omega_3 > 0,$ and $\Omega_1\Omega_2 - \Omega_3 > 0$ [8].

Where

$$\Omega_1 = -\left(a_{11} + a_{22} + a_{33}\right)$$

$$\Omega_2 = \left(a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{12}a_{21} - a_{13}a_{31} - a_{23}a_{32}\right)$$

$$\Omega_3 = -\left(a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} + a_{11}a_{22}a_{33} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}\right)$$

Proof:

The stability analysis around the interior equilibrium point E^* is determined by using the Jacobian matrix as follows:

$$\mathbf{J}_{E^*} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad (24)$$

where

$$a_{11} = N^0 - 2aN^* - \alpha_1 C^* + \frac{\beta_2 N^* P^*}{\left(1+N^*\right)^2} - \frac{\beta_2 P^*}{\left(1+N^*\right)},$$

$$a_{12} = -\alpha_1 N^* + \omega\gamma_1,$$

$$a_{13} = \omega\gamma_2 - \frac{\beta_2 N^*}{1+N^*}$$

$$a_{21} = \alpha_2 C^*$$

$$a_{22} = \alpha_2 N^* + \frac{\beta C^* P^*}{\left(1+C^*\right)^2} - \frac{\beta P^*}{\left(1+C^*\right)} - d_1,$$

$$a_{23} = -\frac{\beta C^*}{1+C^*},$$

$$a_{31} = -\frac{\beta_3 N^* P^*}{\left(1+N^*\right)^2} + \frac{\beta_3 P^*}{\left(1+N^*\right)},$$

$$a_{32} = -\frac{\beta_1 C^* P^*}{\left(1+C^*\right)^2} + \frac{\beta_1 P^*}{\left(1+C^*\right)},$$

$$a_{33} = \frac{\beta_1 C^*}{1+C^*} + \frac{\beta_3 N^*}{1+N^*} - d_2.$$

The corresponding eigenvalues are obtained from the determinant

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0$$

The polynomial equation obtained is given by

$$\lambda^3 + \Omega_1 \lambda^2 + \Omega_2 \lambda + \Omega_3 = 0. \tag{25}$$

By using the Routh-Hurwitz Criteria [9], the positive equilibrium point $E^*\left(N^*, C^*, P^*\right)$ is locally asymptotically stable if and only $\Omega_1 > 0,\ \Omega_3 > 0,$ and $\Omega_1\Omega_2 - \Omega_3 > 0,$ hold.

IV. NUMERICAL SIMULATIONS

In order to illustrate some of the analytical results of the study, numerical simulations of model system (2) are performed using the set of estimated reasonable parameter
$N^0 = 2.5,\ a = 0.26,\ \alpha_1 = 1,\ \alpha_2 = 10,\ \gamma_1 = 0.2,\ \gamma_2 = 0.15,\ \beta = 1.2,\ \beta_1 = 0.2,\ \beta_2 = 0.6,\ \beta_3 = 0.8,$
$d_1 = 0.215,\ d_2 = 0.107,\ \omega = 0.5$

Figures 4.1-4.3 show the phase portraits between the proportion of Nutrient population against Consumer Population, Nutrient population against Predator population and Consumer population against Predator population. This shows the dynamic behaviour of the interior equilibrium of the model system (2) using the parameter values above for different initial starting values in three cases as shown below.

1. $N(0) = 0.4, C(0) = 0.02 \quad and \quad P(0) = 0.1$
2. $N(0) = 0.04, C(0) = 0.004 \quad and \quad P(0) = 0.5$
3. $N(0) = 0.2, C(0) = 0.015 \quad and \quad P(0) = 0.5$

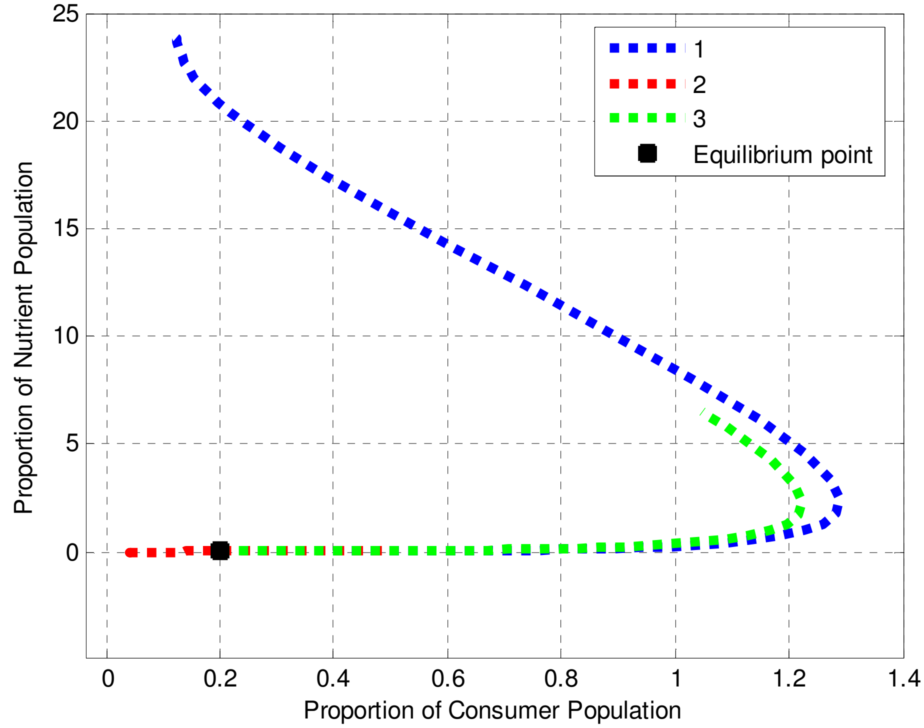


Figure 4.1: **Variation of proportion of Nutrient population against proportion of Consumer population.**

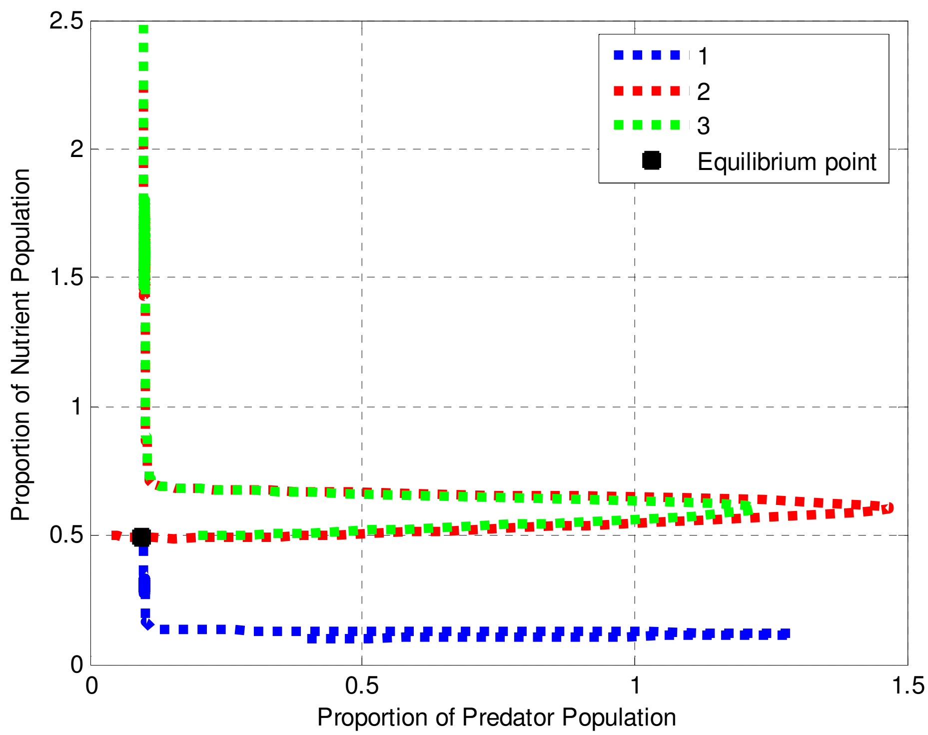


Figure 4.2: **Variation of proportion of Nutrient population against proportion of Predator population.**

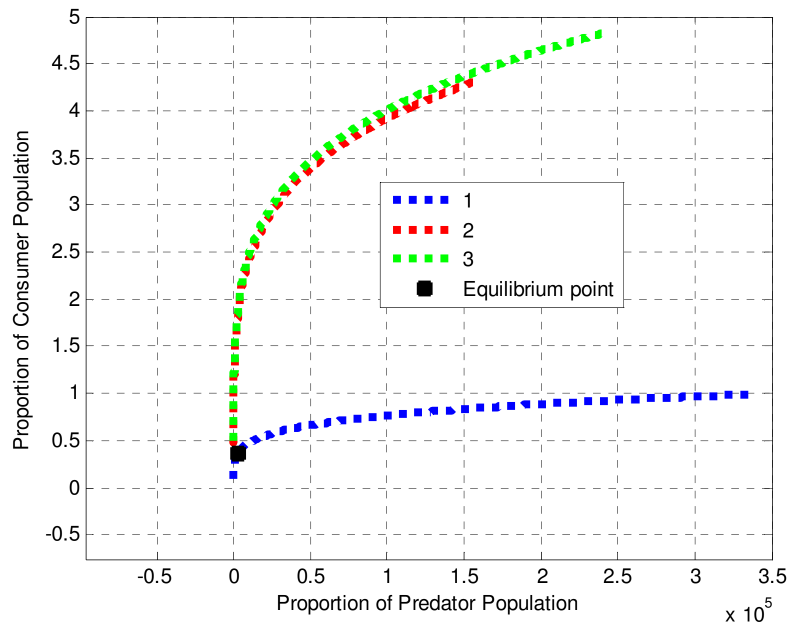


Figure 4.3: **Variation of proportion of Consumer population against proportion of Predator population.**

The equilibrium point of the interior equilibrium E^* was obtained as
$C^* = 0.2 \quad and \quad N^* = 0.015$, $P^* = 0.09422 \quad and \quad N^* = 0.4912$,
$P^* = 3058 \quad and \quad C^* = 0.3673$.

It is observed from these figures that for any starting initial values the solution curves tend to the equilibrium E^*. Therefore we conclude that the model system (2) is globally stable about this interior equilibrium point E^* for parameters displayed above.

Figure 4.4 below shows the variation of the proportion of total population in all classes.

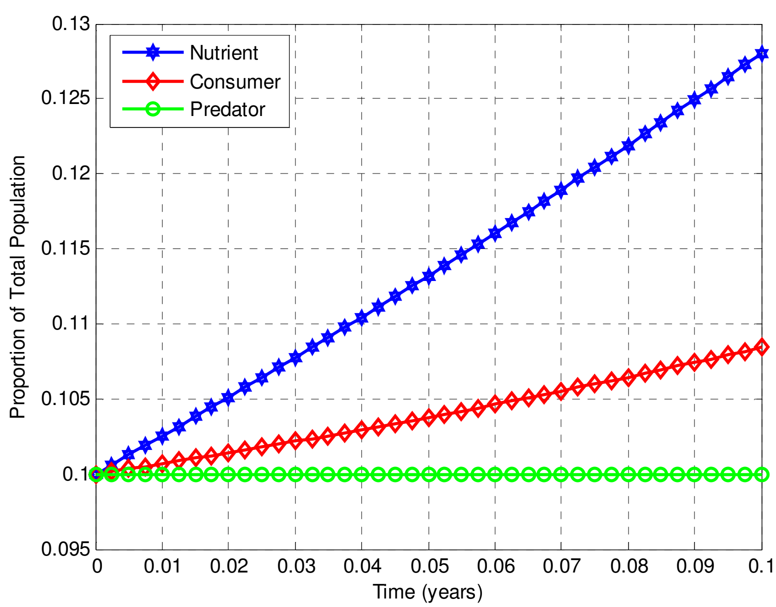


Figure 4.4: **Variation of proportion of Total population in different classes.**

From figure 4.4, it is observed that the proportion of nutrient and consumers increases with time while the predator population is unchanged. This is due to the fact that, as the predator population remains constant, then they don't consume large proportion of nutrients and consumers, and hence the population of nutrients and consumers increases.

Figure 4.5 shows the variation of proportion of predator population growth for different values of $b = N^0$ (constant rate of nutrient supply in the system).

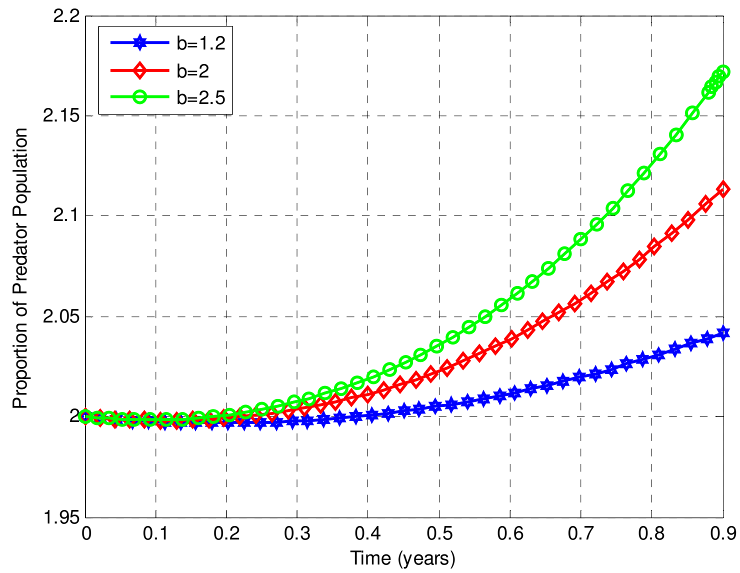


Figure 4.5: **Variation of proportion of Predator population for different values of** b.

From figure 4.5, it is observed that, as values of b increases, the proportion of predator population increases exponentially with time because predators get sufficient food (nutrients) for survival.

Figure 4.6 shows the variation of proportion of nutrient population growth for different values of β_1 (conversion rates of consumer to predator species).

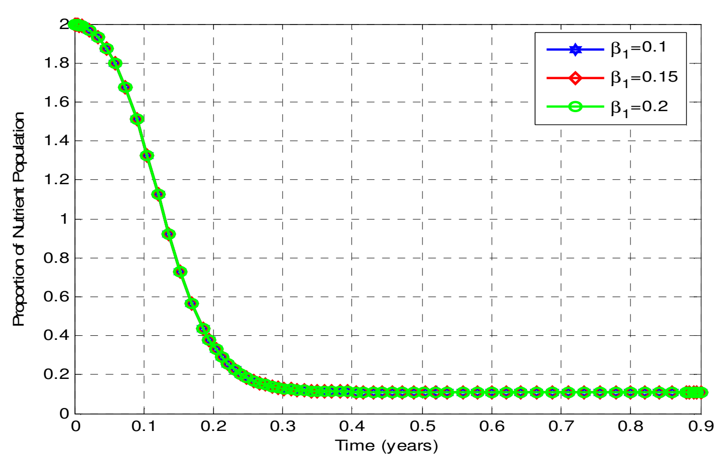


Figure 4.6: **Variation of proportion of Nutrient population for different values** β_1.

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It is seen from figure 4.6 that the increase of the proportion of consumer and predator population leads to the decrease of the amount of nutrient. Therefore, consumers and predators compete for the same resource which is nutrient and as a result, as the number of consumers and predators increases the amount of nutrient decreases and reaches steady state.

Figure 4.7 shows the variation of proportion of consumer population growth for different values of β_3 (conversion rates of nutrient to predator species).

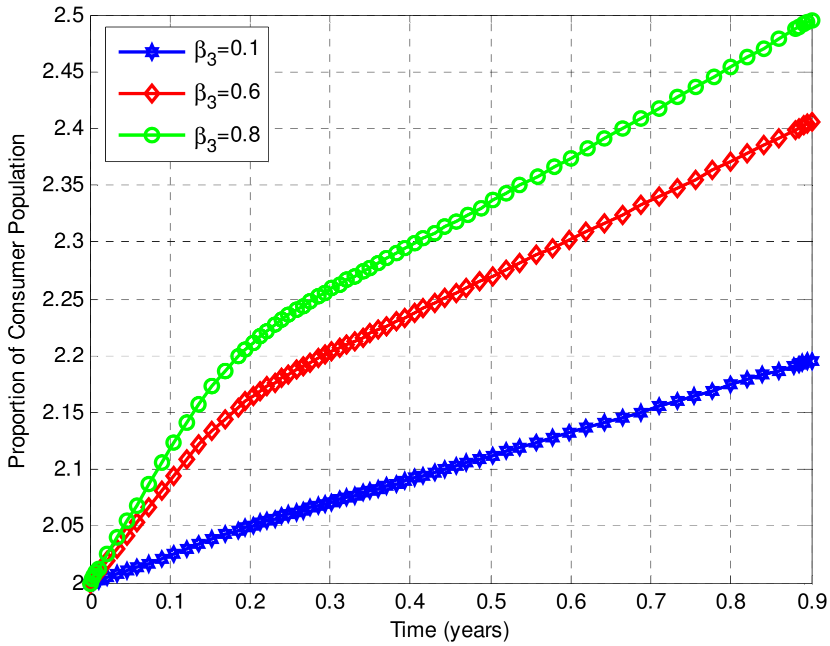

Figure 4.7: Variation of proportion of Consumer population for different values of β_3.

It is observed that the increase of the amount of nutrient and predator leads to the increase of consumer population. This occurred due to the fact that, as predators feed on both nutrients and consumers, in the absence of consumers, predators feed on nutrients and that is why the proportion of consumer population increases.

Figure 4.8 shows the variation of proportion of predator population for different values of α_2 (conversion rates of nutrient supply to consumers).

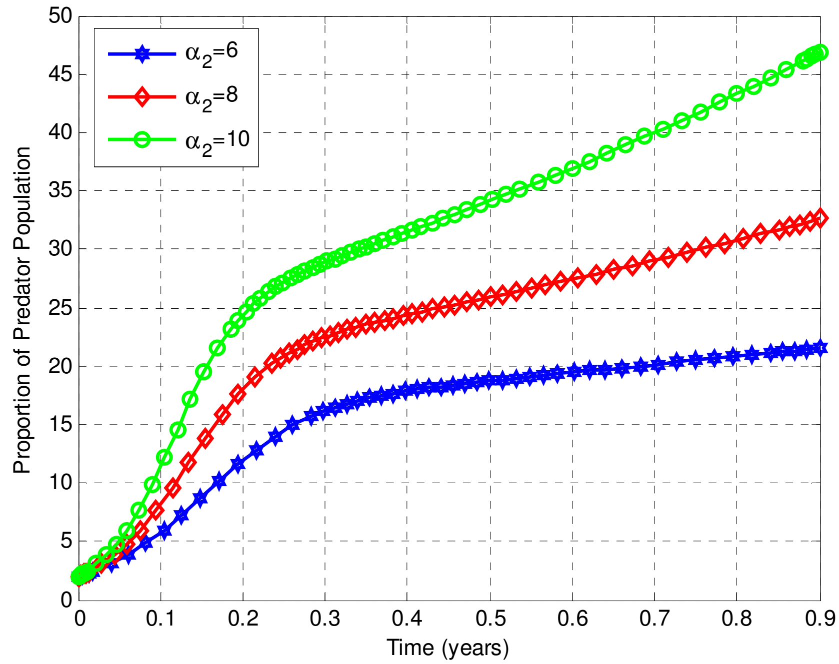

Figure 4.8: Variation of proportion of Predator population growth for different values of α_2.

The increase of the amount of nutrient and consumer species leads to the increase of predator population. The predator population increases because they get sufficient food (i.e. nutrients and consumers). From figure 4.8 above it is seen that as α_2 increases, predator population increases until it reaches steady state.

Figure 4.9 shows the variation of proportion of nutrient population for different values of ω (nutrient regeneration rate from dead consumer and predator population).

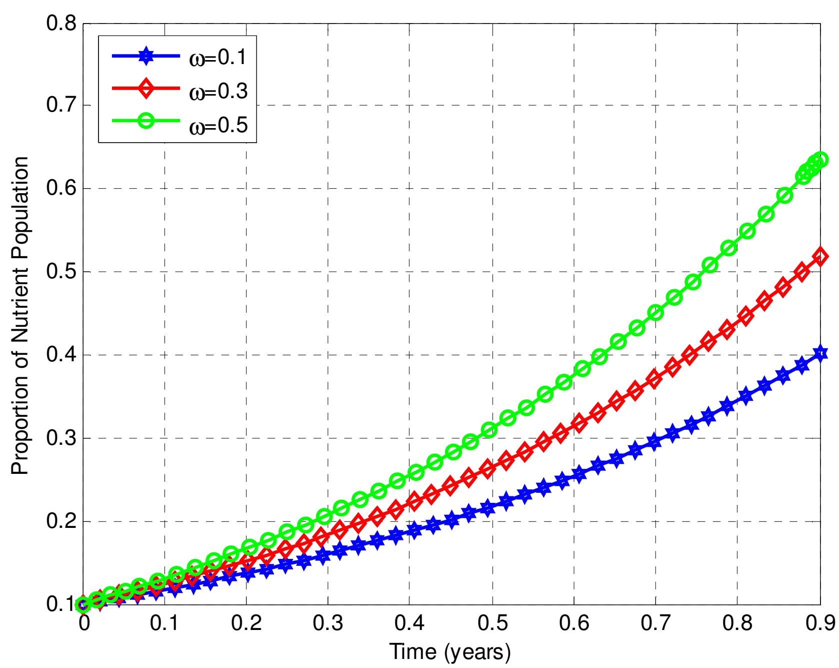

Figure 4.9: **Variation of proportion of Nutrient population for different values of ω.**

It is observed that, when consumer and predator species dies, they regenerate nutrient in the soil. From figure 4.9 it is seen that as ω increases, the proportion of nutrient increases exponentially with time until it reaches the equilibrium point.

Figure 4.10 shows the variation proportion of consumer population growth for different values of $b = N^0$ (constant rate of nutrient supply in the system).

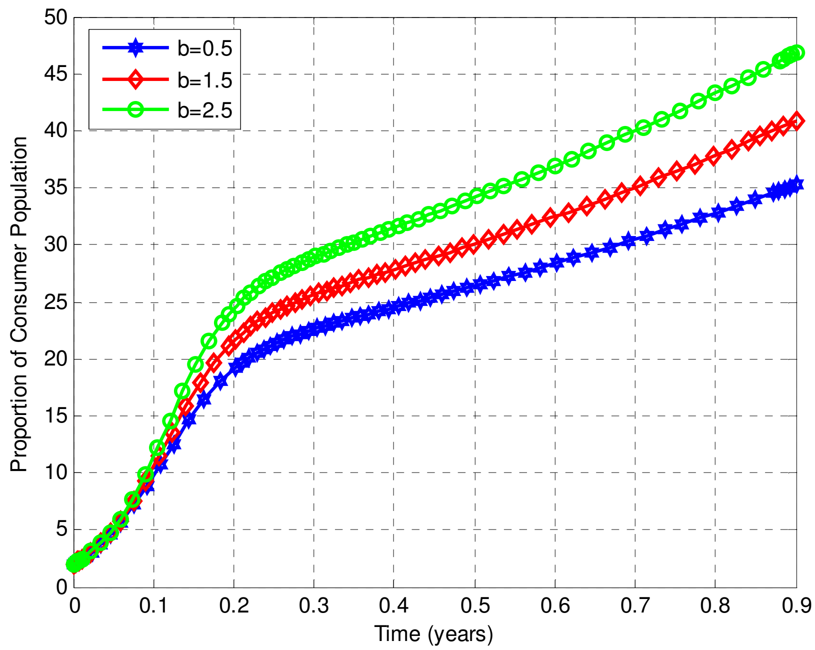

Figure 4.10: Variation of proportion of Consumer population for different values of b.

It is observed that, when values of b increases, the proportion of consumer population increases logistically with time until it reaches steady state because consumers get sufficient food (nutrients) for survival.

V. SUMMARY AND CONCLUSION

In this paper, a nonlinear mathematical model for nutrient, consumer and predator interaction was proposed and analysed to investigate the dynamical behaviour of the system (2). The model incorporates the dependence of predator on both nutrients and consumers. The boundedness and positivity of solutions of the system was derived. Also the existence and local stability of biological equilibrium points were studied and the results were shown. The trivial and predator-free equilibrium points were found to be stable, the axial equilibrium point was found to be stable whenever $\frac{N^0\alpha_2}{a} < d_1$, $\frac{N^0\beta_3}{a+N^0} < d_2$ and unstable otherwise. The interior equilibrium point was investigated by using Routh-Hurwitz criteria and it was found to be stable.

Numerical simulations of the model (2) were performed to determine the impact of the key embedded parameters of the model. From the results of this study, the following conclusions can be drawn: Increasing consumer population and the amount of nutrient and predator, the consumer population also increases. Also as consumer and predator population increase the amount of nutrient decreases. This is due to competition as both consumers and predators compete for the same resource which is nutrients in order to survive. Moreover, wherever the amount of nutrient and consumer increases, predator population also increases. Furthermore, when the rate of nutrients supply in the system increases, the consumer and predator population increases.

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