ONTERNARY QUADRATIC DIOPHANTINE EQUATION

2x^2 + 3y^2 = 4z

M.A. Gopalan¹, K. Geetha² and Manju Somanath³

¹Professor, Dept. of Mathematics, Shrimati Indira Gandhi College, Trichy-02 Tamilnadu, ²Asst Professor, Dept. of Mathematics, Cauvery College for Women, Trichy-18, Tamilnadu, ³Assistant Professor, Dept. of Mathematics, National College, Trichy-01, Tamilnadu, India

Abstract - The ternary quadratic diophantine equation 2x² + 3y² = 4z is analyzed for its non-zero distinct integral points on it. A few interesting properties among the solutions are presented.

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I. INTRODUCTION

Diophantine equations is an interesting concept, as it can be seen from [1-2]. For an extensive review of various problems one may refer [3-11]. In this context one may also see [12-23]. This communication concerns with yet another interesting ternary quadratic diophantine equation 2x² + 3y² = 4z for determining its infinitely many non-zero integral solutions. Also a few interesting properties among the solutions are presented.

II. METHOD OF ANALYSIS

The ternary quadratic equation to be solved for its non-zero integral solution is

2x² + 3y² = 4z      (1)

Assume

x = 2X, y = 2Y      (2)

On substituting (2) in (1), we get

2X² + 3Y² = z      (3)

Let

X = α + 3β, Y = α - 2β, z = 5γ²      (4)

Substituting (4) in (3), we get

α² + 6β² = γ²      (5)

Where α, β and γ are non-zero integers.
Different patterns of solution for (1) are given below

**PATTERN: 1**

Equation (5) can be written as
\[ \alpha^2 + 6\beta^2 = \gamma^2 \times 1 \] (6)

Write 1 as
\[ 1 = \left( \frac{(1+i2\sqrt{6})(1-i2\sqrt{6})}{25} \right) \] (7)

Assume
\[ \gamma = a^2 + 6b^2 \] (8)

Where \(a, b\) are non-zero distinct integers.

Substituting (7) & (8) in (6), we get
\[ \alpha^2 + 6\beta^2 = (a^2 + 6b^2)^2 \left[ \frac{(1+i2\sqrt{6})(1-i2\sqrt{6})}{25} \right] \]

On employing the method of factorization and on equating real and imaginary parts, we get
\[ (a + i\sqrt{6}b)(a - i\sqrt{6}b) = (a + i\sqrt{6}b)^2 \left( \frac{(1+i2\sqrt{6})(1-i2\sqrt{6})}{25} \right) \]

On comparing the positive and negative factors, we get
\[ a + i\sqrt{6}b = (a + i\sqrt{6}b)^2 \left( \frac{1+i2\sqrt{6}}{5} \right) \] (9)
\[ a - i\sqrt{6}b = (a - i\sqrt{6}b)^2 \left( \frac{1-i2\sqrt{6}}{5} \right) \] (10)

On comparing the rational and irrational parts from the above equation, we get
\[ a = \frac{1}{5}(a^2 - 6b^2 - 24ab) \] (11)
\[ \beta = \frac{1}{5}[2(a^2 - 6b^2 + ab)] \] (12)

As our interest is to find only integer solution, it is seen that \(a, \beta\) are integers for suitable choices of \(a\) and \(b\).

Let us assume \(a = 5A\) and \(b = 5B\) in (11) and (12), we get
\[ \alpha = 5A^2 - 30B^2 - 120AB \] (13)
\[ \beta = 10A^2 - 60B^2 + 10AB \] (14)

Substituting (13) and (14) in (2), thenon-zero distinct integral solutions of (1), we get
\[ x = x(A, B) = 2 \left( 35A^2 - 210B^2 - 90AB \right) \]
\[ y = y(A, B) = 2 \left( -15A^2 + 90B^2 - 140AB \right) \]
\[ z = z(A, B) = 5 \left( 25A^2 + 150B^2 \right)^2 \]

**PROPERTIES:**

1. \[ x(A,1) + y(A,1) - t_{42,A} - 4t_{12,A} \equiv -240 \text{(mod 425)} \]

2. \[ z(A,1) = 250 \left( c_{25,A}^2 + 55c_{5,A} \right) - 5 \left( 3437g_A + A + 16263 \right) \]

**PATTERN: 2**

Write (5) as,
\[ \gamma^2 - \alpha^2 = 6\beta^2 \]
\[(\gamma + \alpha) (\gamma - \alpha) = 6\beta^2 \quad (15)\]

**Case 1:**

\[
\frac{\gamma + \alpha}{6\beta} = \frac{\beta}{\gamma - \alpha} = \frac{p}{q} \quad (16)
\]

This is equivalent to the following two equations

\[
q\alpha - 6p\beta + q\gamma = 0
\]

\[
p\alpha + q\beta - p\gamma = 0
\]

Applying the method of cross multiplication, we get

\[
\begin{aligned}
\alpha &= \alpha(p, q) = 6p^2 - q^2 \\
\beta &= \beta(p, q) = 2pq \\
\gamma &= \gamma(p, q) = 6p^2 + q^2
\end{aligned} \quad (17)
\]

Substituting the values of (17) in (2), the non-zero distinct integral values of x, y and z satisfying (1) are given by

\[
\begin{aligned}
x &= x(p, q) = 2(6p^2 - q^2 + 6pq) \\
y &= y(p, q) = 2(6p^2 - q^2 - 4pq) \\
z &= z(p, q) = 5(36p^4 + q^4 + 12p^2q^2)
\end{aligned}
\]

**PROPERTIES:**

1. \(x(1, q) - y(1, q) + z(1, q) = ct_{10, q^2} + t_{112, q} + 37g, q + 216\)

2. \(x(n, 2n+1) - y(n, n+1) - 4ln = t_{54, n}\)

**Case 2:**

Equation (15) can be rewritten as

\[
\frac{\gamma + \alpha}{\beta} = \frac{6\beta}{\gamma - \alpha} = \frac{p}{q}
\]

On following the procedure as in case (1) the non-zero distinct solutions of (1) are given by

\[
\begin{aligned}
x &= x(p, q) = 2(p^2 - 6q^2 + 6pq) \\
y &= y(p, q) = 2(p^2 - 6q^2 - 4pq) \\
z &= z(p, q) = 5(p^4 + 36q^4 + 12p^2q^2)
\end{aligned}
\]

**PROPERTIES:**

1. \(-\frac{x(2^n, n)}{2} + Mer_{2n} + 6\omega_n + 7 = \text{Nasty number}\)

2. \(x(2^n, 2^n) - y(2^n, 2^n) = 10(car_{1n} + ky_n) + 20\)

**PATTERN: 3**

Rewrite (5) as,

\[
\gamma^2 - 6\beta^2 = \alpha^2 \quad (18)
\]

Write 1 as,

\[
1 = (5 + 2\sqrt{6})(5 - 2\sqrt{6}) \quad (19)
\]

Assume

\[
\alpha = a^2 - 6b^2 \quad (20)
\]

Using (19) and (20) in (18), we get
\[ \gamma^2 - 6\beta^2 = \left( a^2 - 6b^2 \right)^2 \left( 5 + 2\sqrt{6} \right) \left( 5 - 2\sqrt{6} \right) \]

On employing the method of factorization and equating the positive and negative factors, we get
\[
\begin{align*}
\alpha &= a^2 - 6b^2 \\
\beta &= 2a^2 + 12b^2 + 10ab \\
\gamma &= 5a^2 + 30b^2 + 24ab
\end{align*}
\]

Substituting the values of (21) in (2), the non-zero distinct integral values of \(x, y, \) and \(z\) satisfying (1) are given by
\[
\begin{align*}
x &= x(a, b) = 2 \left( 7a^2 + 30b^2 + 30ab \right) \\
y &= y(a, b) = 2 \left( -3a^2 - 30b^2 - 20ab \right) \\
z &= z(a, b) = 5 \left( 5a^2 + 30b^2 + 24ab \right)^2
\end{align*}
\]

**PROPERTIES:**

1. \(x(a+1,1) + y(a+1,1) - (t_{8,a} + t_{12,a} + 21g_a) = 49\)
2. \(y(n+3,-2n) = 2 \left( -t_{68,n} - t_{62,n} - t_{42,n} + 11g_n - 16 \right)\)

**III. CONCLUSION**

One may search for other patterns of solution and their corresponding properties.

**REFERENCES**


[22] Gopalan M.A., Geetha.K and Manju Somanath , “A ternary quadratic Diophantine equation \( 8\left(x^2 + y^2\right) - 15xy + \left(x + y\right) + 1 = 32z^2 \),” Proceedings of the international conference on Mathematical methods and computation, Feb 13th and 14th 2014, 246- 251.