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Degree of Approximation of Signals (Functions) by Product Means BA of Fourier Series in Weighted Lipschitz Class- $W(L^p,\omega(t),\beta)$ Approximation of $f \in W(L^p,\omega(t),\beta)$ by Product Means

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Abstract—In this paper, we estimate the uniform approximation of periodic functions belonging to $f \in W(L^p,\omega(t),\beta)$ class, where ω is an increasing function associated with f, using product means of the Fourier series of f. We also discuss the case $p=1$ separately. The deviation obtained in our theorems is free from p and sharper than the earlier results.

Keywords- $f \in W(L^p,\omega(t),\beta)$; Degree of approximation; Product means; Fourier series: Summability.

I. INTRODUCTION

For a given 2π - periodic function $f \in L^p = L^p[0,2\pi], p \geq 1$, let

$$s_n(f;x) := a_0/2 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx), n \in \mathbb{N}, \text{ with } s_0(f;x) = a_0/2 \qquad (1)$$

denote the n^{th} partial sum , called trigonometric polynomial of degree (or order) n, of the Fourier series of f.

Let $T \equiv (a_{n,k})$ be a lower triangular matrix. Then the sequence to sequence transformation

$$t_n(f;x) = \sum_{k=0}^{n} a_{n,k} s_k(f;x), n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},$$

defines the matrix means of $\{s_n(f;x)\}$. The Fourier series (1) is said to be summable to s by T - means, if $\lim_{n\to\infty} t_n(f;x) = s$, where s is a finite number [6].

Let $A \equiv (a_{n,m})$ and $B \equiv (b_{n,m})$ be infinite lower triangular matrices of real numbers such that

$$A(\text{or } B) = \begin{cases} a_{n,m}(\text{or } b_{n,m}) \geq 0, & m = 0,1,2,\ldots n, \\ a_{n,m}(\text{or } b_{n,m}) = 0, & m > n, \end{cases} \qquad (2)$$

$$\sum_{m=0}^{n} a_{n,m} = 1 \text{ and } \sum_{m=0}^{n} b_{n,m} = 1, \text{ where } n = 0,1,2,\ldots, \qquad (3)$$

and let $A_{n,r} = \sum_{m=0}^{r} a_{n,m}$ and $\bar{A}_{n,r} = \sum_{m=r}^{n} a_{n,m}, B_{n,r} = \sum_{m=0}^{r} b_{n,m}$ and $\bar{B}_{n,r} = \sum_{m=r}^{n} b_{n,m},$ (4)

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so that $A_{n,n} = B_{n,n} = 1 = \bar{A}_{n,0} = \bar{B}_{n,0}$. (5)

When we superimpose the $B-$ summability on $A-$ summability, we get $B.A$ means of $\{s_k(f;x)\}$ defined by

$$t_n^{B.A}(f;x) = \sum_{m=0}^{n} b_{n,m}\left(\sum_{k=0}^{m} a_{m,k} s_k(f;x)\right) = \sum_{m=0}^{n}\sum_{k=0}^{m} b_{n,m} a_{m,k} s_k(f;x),\ n = 0,1,2,\dots. \tag{6}$$

We write $(B.A)_n(t)$ as

$$(B.A)_n(t) = \frac{1}{2\pi}\sum_{m=0}^{n}\sum_{k=0}^{m} b_{n,m} a_{m,k} \frac{\sin(k+1/2)t}{\sin(t/2)}$$

and we also write $\phi(t) \equiv \phi(x,t) := f(x+t) - f(x-t)$.

The L^p - norm of f is defined by

$$\|f\|_p = \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^p\, dx\right\}^{1/p} \quad (1 \le p < \infty), \text{ and } \|f\|_\infty = \operatorname{ess\,sup}_{x\in[0,2\pi]} |f(x)|.$$

A function $f \in W(L^p, \omega(t), \beta)$ if $\left(\int_0^{2\pi} \left|\left(f(x+t) - f(x)\right)\sin^\beta(x/2)\right|^p dx\right)^{1/p} = O\left(\omega(t)\right)$ where $\beta \ge 0,\ p \ge 1$ and $\omega(t)$ is a positive increasing function of β and t.

If $\beta = 0,\ W(L^p, \omega(t), \beta) \equiv Lip(\xi(t), p)$ and for $\xi(t) = t^\alpha\ (0 < \alpha \le 1),\ Lip(\xi(t), p) \equiv Lip(\alpha, p)$.

$Lip(\alpha, p) \to Lip\alpha$ as $p \to \infty$. Thus $Lip\alpha \subseteq Lip(\alpha, p) \subseteq Lip(\xi(t), p) \subseteq W(L^p, \xi(t))$ [1,3,4 and references therein].

We write $K_1 \ll K_2$ if $\exists$ a positive constant C (it may depend on some parameters), such that $K_1 \le CK_2$.

II. KNOWN RESULTS

The product means have been of growing interests over the last few years, many investigators considered product means in various directions for example Lal [1] used $C^1.N_p$ means of Fourier series in weighted class and proved the following result:

$$\left\|t_n^{CN} - f\right\|_r = O\left((n+1)^{\beta+1/r}\,\xi\left(1/(n+1)\right)\right)$$

In the subsequent results Mishra et al. [3] proved his results using $C^1.N_p$ means of conjugate series of Fourier series in the same class.

Lenski and Szal [2] have proved their results using product means, BA, and obtained pointwise approximation in terms of integral modulus of continuity.

Further, very recently, Singh and Srivastava [4, Theorem 2.2, p. 4] obtained the degree of approximation of functions belonging to weighted Lipschitz class $W(L^p, \xi(t), \beta)$ by $C^1.T$ means of its Fourier series and the result is given as

$$\| t_n^{C^1.T}(f;x) - f(x)\|_p = O((n+1)^\beta \omega(1/(n+1))),$$

where $\omega(t)$ is a positive increasing function. We note that deviation in this result is free from p.

Remark 1. *Recently, Zhang* [5] *have pointed out that the results of Mishra et al.* [3] *only hold for the constant functions under their assumption conditions. The authors have assumed the following conditions*

1. $\xi(t)/t$ is decreasing function,

2. $\left\{\int_0^{\pi/(n+1)}\left(|\psi_x(t)|\sin^{\beta}(t/2)/\xi(t)\right)^p dt\right\}^{1/p}=O(1),$

3. $\left\{\int_{\pi/(n+1)}^{\pi}\left(t^{-\delta}|\psi_x(t)|\sin^{\beta}(t/2)/\xi(t)\right)^p dt\right\}^{1/p}=O\left((n+1)^{\delta}\right),$

where δ *is an arbitrary number such that* $(\beta-\delta)s-1>0,\ \ p^{-1}+q^{-1}=1.$

The author [5] *has shown that* $\beta<(1-1/r)$ *and so* $\delta<(\beta-1/s)<0$. *From assumption 2,* $\psi_x(t)$ *becomes constant and, hence, the results are trivial. Similarly error can also be seen in* [1]. *These observations motivated us to study the problem further so that these errors can be removed.*

III. MAIN RESULTS

We extend the earlier results in such a way that the above means has been replaced by more general product means of Fourier series generated by the product of two general matrices. We give our results in view Remark 1. More precisely, we prove the following theorem:

Theorem 1. *Let* $f\in W\left(L^p,\omega(t),\beta\right)$ *with* $0<\beta<1-1/p,\ p>1,$ *and the entries of the lower triangular matrices* $A\equiv(a_{n,k})$ *and* $B\equiv(b_{n,k})$ *satisfy the following conditions:*

$$b_{n,n}\asymp 1/(n+1), \tag{7}$$

$$|b_{n,m}a_{m,0}-b_{n,m+1}a_{m+1,1}|\ll b_{n,m}/(m+1)^2 \quad \text{for } 0\le m\le n-1 \tag{8}$$

$$\sum_{k=0}^{m-1}|(b_{n,m}a_{m,m-k}-b_{n,m+1}a_{m+1,m+1-k})-(b_{n,m}a_{m,m-k-1}-b_{n,m+1}a_{m+1,m-k})|\ll\frac{b_{n,m}}{(m+1)^2} \quad \text{for } 0\le m\le n-1, \tag{9}$$

with $A_{n,n}=B_{n,n}=1$ *for* $n=0,1,2,\ldots$. *Then the degree of approximation of* f *by* BA *means of its Fourier series is given by*

$$\left|t_n^{B.A}(f;x)-f(x)\right|=O\left(\sum_{m=0}^{n}\frac{b_{n,m}}{m+1}(n+1)^{\beta+1}\omega(1/(n+1))\right),$$

provided that the positive non-decreasing function ω, *satisfies the following conditions:*

$$\omega(t)/t \text{ is decreasing function,} \tag{10}$$

$$\left\{\int_0^{\pi/(n+1)}\left(|\phi(t)|\sin^{\beta}(t/2)/\omega(t)\right)^p dt\right\}^{1/p}=O\left((n+1)^{-1/p}\right), \tag{11}$$

$$\left\{\int_{\pi/(n+1)}^{\pi}\left(t^{-\delta}|\phi(t)|\sin^{\beta}(t/2)/\omega(t)\right)^p dt\right\}^{1/p}=O\left((n+1)^{\delta-1/p}\right), \tag{12}$$

where δ *is an arbitrary number such that* $1/p<\delta<(\beta+1/p)$, $p^{-1}+q^{-1}=1$ *and conditions* (11) *and* (12) *hold uniformly in* x.

Note 1. *Conditions* (10) *implies that*

$$\omega(\pi/(n+1))/\left(\pi/(n+1)\right)\le\omega(1/(n+1))/\left(1/(n+1)\right) \text{ i.e., } \omega\left(\pi/(n+1)\right)=O\left(\omega\left(1/(n+1)\right)\right).$$

IV. LEMMAS

We need the following lemmas for proving our theorem.

Lemma 1.*If the conditions and hold, then*

$$|b_{n,r}a_{r,r-l}-b_{n,r+1}a_{r+1,r+1-l}| \ll b_{n,r}/(r+1)^2, \quad for\ 0\le l\le r-l\le n-2.$$

For the proof one can see [3, Lemma 3.2, p.1123].

Lemma 2. *If the matrices* A *and* B *satisfy the conditions of Theorem* 1, *then*

$$|(B.A)_n(t)|=O(n+1),\ for\ 0<t\le\pi/(n+1).$$

Proof: Using $1/\sin(t/2)=O(\pi/t)$ and $0\le sin(nt)\le nt$, for $0<t\le\pi/(n+1)$, we have

$$\begin{aligned}|(B.A)_n(t)| &= \left|\frac{1}{2\pi}\sum_{m=0}^{n}\sum_{k=0}^{m}b_{n,m}a_{m,k}\frac{\sin(k+1/2)t}{\sin(t/2)}\right|\\ &\le \frac{1}{2\pi}\sum_{m=0}^{n}\sum_{k=0}^{m}b_{n,m}a_{m,k}\left|\frac{\sin(k+1/2)t}{\sin(t/2)}\right| = O\left(\sum_{m=0}^{n}\sum_{k=0}^{m}b_{n,m}a_{m,k}\frac{(k+1)t}{t}\right)\\ &= O\left((n+1)\sum_{m=0}^{n}b_{n,m}\left(\sum_{k=0}^{m}a_{m,k}\right)\right)=O\left((n+1)\sum_{m=0}^{n}b_{n,m}A_{m,m}\right)\\ &= O\left((n+1)B_{n,n}\right)=O(n+1),\end{aligned}$$

in view of $A_{n,n}=B_{n,n}=1$.

Lemma 3. *If the matrices* A *and* B *satisfy the conditions of Theorem* 1, *then*

$$|(B.A)_n(t)|=O\left(\frac{1}{t^2}\left(\sum_{m=0}^{n}\frac{b_{n,m}}{m+1}+\frac{1}{n+1}\right)\right),\ for\ \pi/(n+1)<t\le\pi.$$

Proof: Using $1/\sin(t/2)=O(\pi/t)$ for $\pi/(n+1)<t\le\pi$, we have

$$|(B.A)_n(t)| = \left|\frac{1}{2\pi}\sum_{m=0}^{n}\sum_{k=0}^{m}b_{n,m}a_{m,k}\frac{\sin(k+1/2)t}{\sin(t/2)}\right| = O(1/t)\left|\sum_{m=0}^{n}\sum_{k=0}^{m}b_{n,m}a_{m,k}\sin(k+1/2)t\right|.$$

Now, using Abel's transformation after changing the order of summation, we have

$$\begin{aligned}\left|\sum_{m=0}^{n}\sum_{k=0}^{m}b_{n,m}a_{m,k}\sin(k+1/2)t\right| &= \left|\sum_{m=0}^{n}\sum_{k=0}^{m}b_{n,m}a_{m,m-k}\sin(m-k+1/2)t\right|\\ &= \left|\sum_{k=0}^{n}\left[\sum_{m=k}^{n-1}(b_{n,m}a_{m,m-k}-b_{n,m+1}a_{m+1,m+1-k})\sum_{l=m}^{k}\sin(l-k+1/2)t+b_{n,n}a_{n,n-k}\sum_{l=k}^{n}\sin(l-k+1/2)t\right]\right|\\ &= O(1/t)\left(\sum_{m=0}^{n-1}\left[\sum_{k=0}^{m}|b_{n,m}a_{m,m-k}-b_{n,m+1}a_{m+1,m+1-k}|\right]+\sum_{k=0}^{n}b_{n,n}a_{n,n-k}\right)\\ &= O(1/t)\left[\sum_{m=0}^{n-1}m.\frac{b_{n,m}}{(m+1)^2}+b_{n,n}+\sum_{m=0}^{n-1}\frac{b_{n,m}}{(m+1)^2}\right]=O(1/t)\left[\sum_{m=0}^{n}\frac{b_{n,m}}{(m+1)}+\frac{1}{(n+1)}\right],\end{aligned}$$

in view of Lemma 1 conditions (7) and (9) and $A_{n,n}=1$.

Hence

$$|(B.A)_n(t)|=O\left(\frac{1}{t^2}\left(\sum_{m=0}^{n}\frac{b_{n,m}}{m+1}+\frac{1}{n+1}\right)\right).$$

V. PROOF OF MAIN RESULT

4.1 Proof of Theorem 1: By using the integral representation of $s_k(f:x)$ we have

$$s_k(f;x)-f(x)=\frac{1}{2\pi}\int_0^{\pi}\phi(t)\frac{\sin(k+1/2)t}{\sin(t/2)}dt.$$

From (6), we have

$$\begin{aligned}
t_n^{B.A}(f;x)-f(x)&=\sum_{m=0}^{n}\sum_{k=0}^{m}b_{n,m}a_{m,k}(s_k(f;x)-f(x))\\
&=\sum_{m=0}^{n}\sum_{k=0}^{m}b_{n,m}a_{m,k}\left(\frac{1}{2\pi}\int_0^{\pi}\phi(t)\frac{\sin(k+1/2)t}{\sin(t/2)}dt\right)\\
&=\frac{1}{2\pi}\int_0^{\pi}\phi(t)\sum_{m=0}^{n}\sum_{k=0}^{m}b_{n,m}a_{m,k}\frac{\sin(k+1/2)t}{\sin(t/2)}dt=\int_0^{\pi}\phi(t)(B.A)_n(t)dt\\
&=\int_0^{\pi/(n+1)}\phi(t)(B.A)_n(t)dt+\int_{\pi/(n+1)}^{\pi}\phi(t)(B.A)_n(t)dt\\
&=I_1+I_2. \qquad (13)
\end{aligned}$$

Now, using Lemma 2, $1/\sin(t/2)=O(\pi/t)$, for $0<t\le\pi/(n+1)$ and Hölder's inequality, we have

$$\begin{aligned}
|I_1|&\le\int_0^{\pi/(n+1)}\left|\phi(t)(B.A)_n(t)\right|dt=\lim_{\varepsilon\to0}\int_{\varepsilon}^{\pi/(n+1)}|\phi(t)||(B.A)_n(t)|dt\\
&=O\left(\lim_{\varepsilon\to0}\int_{\varepsilon}^{\pi/(n+1)}\frac{|\phi(t)|\sin^{\beta}(t/2)}{\omega(t)}.\frac{(n+1)\,\omega(t)}{\sin^{\beta}(t/2)}dt\right)\\
&=O\left((n+1)\left\{\int_0^{\pi/(n+1)}\left(\frac{|\phi(t)|\sin^{\beta}(t/2)}{\omega(t)}\right)^p dt\right\}^{1/p}\times\left\{\lim_{\varepsilon\to0}\int_{\varepsilon}^{\pi/(n+1)}\left(\frac{\omega(t)}{\sin^{\beta}(t/2)}\right)^q dt\right\}^{1/q}\right)\\
&=O\left[(n+1)^{1-1/p}\omega(\pi/(n+1))\left\{\lim_{\varepsilon\to0}\int_{\varepsilon}^{\pi/(n+1)}t^{-q\beta}dt\right\}^{1/q}\right]\\
&=O\left[(n+1)^{1-1/p}\omega(\pi/(n+1))(n+1)^{\beta-1/q}\right]=O\left(\omega(\pi/(n+1))(n+1)^{\beta}\right), \qquad (14)
\end{aligned}$$

in view of condition (11), mean value theorem for integrals, $0\le\beta<1-1/p$ and $p^{-1}+q^{-1}=1$.
Again, using Lemma 3, $1/\sin(t/2)=O(\pi/t)$, for $0<t\le\pi/(n+1)$ and Hölder's inequality, we have

$$\begin{aligned}
|I_2|&\le\int_{\pi/(n+1)}^{\pi}\left|\phi(t)(B.A)_n(t)\right|dt=\int_{\pi/(n+1)}^{\pi}\frac{|\phi(t)|}{t^2(n+1)}dt+\int_{\pi/(n+1)}^{\pi}\frac{|\phi(t)|}{t^2}\sum_{m=0}^{n-1}\frac{b_{n,m}}{(m+1)}dt\\
&=I_{21}+I_{22}. \qquad (15)
\end{aligned}$$

Now, we have

$$\begin{aligned}
I_{21}&=O\left[(n+1)^{-1}\left\{\int_{\pi/(n+1)}^{\pi}\left(t^{-\delta}\frac{|\phi(t)|\sin^{\beta}(t/2)}{\omega(t)}\right)^p dt\right\}^{1/p}\times\left\{\int_{\pi/(n+1)}^{\pi}\left(\frac{\omega(t)}{t^{-\delta+2}\sin^{\beta}(t/2)}\right)^q dt\right\}^{1/q}\right]\\
&=O\left[(n+1)^{\delta-1/p-1}\left\{\int_{\pi/(n+1)}^{\pi}\left((\omega(t)/t).t^{-(-\delta+\beta+1)}\right)^q dt\right\}^{1/q}\right]
\end{aligned}$$

$$= O\left[(n+1)^{\delta-1/p-1}\omega(\pi/(n+1))(n+1)\left\{\int_{\pi/(n+1)}^{\pi} t^{-q(-\delta+\beta+1)}dt\right\}^{1/q}\right]$$

$$= O\left[(n+1)^{\delta-1/p}\omega(\pi/(n+1))(n+1)^{-\delta+\beta+1-1/q}\right] = O\left(\omega(\pi/(n+1))(n+1)^{\beta}\right), \tag{16}$$

in view of conditions (10) and (12), mean value theorem for integrals, $1/p < \delta < \beta + 1/p$ and $p^{-1} + q^{-1} = 1$.

Similaly, we have

$$I_{22} = O\left[\sum_{m=0}^{n}\frac{b_{n,m}}{(m+1)}\left\{\int_{\pi/(n+1)}^{\pi}\left(\frac{t^{-\delta}\,|\phi(t)|\sin^{\beta}(t/2)}{\omega(t)}\right)^{p}dt\right\}^{1/p} \times \left\{\int_{\pi/(n+1)}^{\pi}\left(\frac{\omega(t)}{t^{-\delta+2}\sin^{\beta}(t/2)}\right)^{q}dt\right\}^{1/q}\right]$$

$$= O\left[\sum_{m=0}^{n}\frac{b_{n,m}}{(m+1)}(n+1)^{\delta-1/p}\left\{\int_{\pi/(n+1)}^{\pi}\left((\omega(t)/t).t^{-(-\delta+\beta+1)}\right)^{q}dt\right\}^{1/q}\right]$$

$$= O\left[\sum_{m=0}^{n}\frac{b_{n,m}}{(m+1)}(n+1)^{\delta-1/p}\omega(\pi/(n+1))(n+1)(n+1)^{-\delta+\beta+1-1/q}\right]$$

$$= O\left(\sum_{m=0}^{n}\frac{b_{n,m}}{(m+1)}\omega(\pi/(n+1))(n+1)^{\beta+1}\right), \tag{17}$$

in view of conditions (10) and (12), mean value theorem for integrals, $1/p < \delta < \beta + 1/p$ and $p^{-1} + q^{-1} = 1$.

Further, we have

$$(n+1)^{\beta}\omega(\pi/(n+1)) + \sum_{m=0}^{n}\frac{b_{n,m}}{m+1}(n+1)^{\beta+1}\omega(\pi/(n+1) = \omega(\pi/(n+1))(n+1)^{\beta}\left[1+\sum_{m=0}^{n}\frac{b_{n,m}}{m+1}(n+1)\right]$$

$$\geq 2\omega(\pi/(n+1))(n+1)^{\beta},$$

that is

$$(n+1)^{\beta}\omega(\pi/(n+1)) = O\left(\sum_{m=0}^{n}\frac{b_{n,m}}{m+1}(n+1)^{\beta+1}\omega(\pi/(n+1)\right). \tag{18}$$

Collecting (13) - (18), we get

$$|t_n^{B.A}(f;x) - f(x)| = O_x\left(\sum_{m=0}^{n}\frac{b_{n,m}}{m+1}(n+1)^{\beta+1}\omega(1/(n+1))\right),$$

in view of Note 1.

This completes the proof of Theorem 1.

Remark 1. *In the proof of above theorem we have used* Hölder's *inequality for* $p > 1$. *Therefore, the proof is not applicable for* $p = 1$. *Moreover for* $p = 1$, β *becomes negative. Thus, for* $p = 1$, *we have the following theorem*:

Theorem 2. *Let* $f \in W\left(L^{1}, \omega(t), \beta\right)$ *with* $0 < \beta < 1$ *and the entries of the lower triangular matrices* A *and* B *satisfy the conditions* (7)-(9) *with* $A_{n,n} = B_{n,n} = 1$ *for* $n = 0,1,2,\ldots$. *Then the degree of approximation of* f *by* BA *means of its Fourier series is given by*

$$\left|t_n^{B.A}(f;x) - f(x)\right| = O\left(\sum_{m=0}^{n}\frac{b_{n,m}}{m+1}(n+1)^{\beta+1}\omega(1/(n+1))\right),$$

provided that the positive non-decreasing function ω, *satisfies the condition* (10) *and the following conditions:*

$$\omega(t)/t^{\beta} \text{ is non-decreasing function,} \tag{19}$$

$$\int_0^{\pi/(n+1)} \left(|\phi(t)|\sin^{\beta}(t/2)/\omega(t)\right)dt = O\left((n+1)^{-1}\right), \tag{20}$$

$$\int_{\pi/(n+1)}^{\pi} t^{-\delta}\left|\phi(t)\right|/\omega(t)dt = O\left((n+1)^{\delta-1}\right), \tag{21}$$

where δ *is an arbitrary number such that* $1<\delta<\beta+1$ *and* $p^{-1}+q^{-1}=1$.

Proof of Theorem 2:

Following the proof of Theorem 1 and Hölder's inequality for $p=1, q=\infty$, we have

$$|I_1| = O\left[(n+1)\int_0^{\pi/(n+1)} \frac{|\phi(t)|\sin^{\beta}(t/2)}{\omega(t)}dt \times \operatorname{ess}\sup_{0<t\le\pi/(n+1)}\left|\frac{\omega(t)}{\sin^{\beta}(t/2)}\right|\right]$$

$$= O\left[(n+1)(n+1)^{-1}\operatorname{ess}\sup_{0<t\le\pi/(n+1)}\left|\frac{\omega(t)}{t^{\beta}}\right|\right] = O\left(\omega(\pi/(n+1))(n+1)^{\beta}\right), \tag{22}$$

in view of conditions (19) and (20) .

Similarly,

$$I_{21} = O\left[(n+1)^{-1}\int_{\pi/(n+1)}^{\pi} \frac{t^{-\delta}|\phi(t)|\sin^{\beta}(t/2)}{\omega(t)}dt \times \operatorname{ess}\sup_{\pi/(n+1)<t\le\pi}\left|\frac{\omega(t)}{t^{-\delta+2}\sin^{\beta}(t/2)}\right|\right]$$

$$= O\left[(n+1)^{\delta-1-1}\omega\left(\frac{\pi}{n+1}\right)\left(\frac{(n+1)^{2+\beta-\delta}}{\pi^{2+\beta-\delta}}\right)\right] = O\left(\omega(\pi/(n+1))(n+1)^{\beta}\right), \tag{23}$$

in view of decreasing nature of $\omega(t)/t^{\beta+2-\delta}$ and condition (\ref{ch1:29}).

$$I_{22} = O\left[\sum_{m=0}^{n}\frac{b_{n,m}}{(m+1)}\int_{\pi/(n+1)}^{\pi} \frac{t^{-\delta}|\phi(t)|\sin^{\beta}(t/2)}{\omega(t)}dt \times \operatorname{ess}\sup_{\pi/(n+1)<t\le\pi}\left|\frac{\omega(t)}{t^{-\delta+2}\sin^{\beta}(t/2)}\right|\right]$$

$$= O\left[\sum_{m=0}^{n}\frac{b_{n,m}}{(m+1)}(n+1)^{\delta-1}\omega\left(\frac{\pi}{n+1}\right)\left(\frac{(n+1)^{2+\beta-\delta}}{\pi^{2+\beta-\delta}}\right)\right]$$

$$= O\left(\sum_{m=0}^{n}\frac{b_{n,m}}{(m+1)}\omega(\pi/(n+1))(n+1)^{\beta+1}\right), \tag{24}$$

in view of decreasing nature of $\omega(t)/t^{\beta+2-\delta}$ and condition (21).

Collecting (22)-(24), we get

$$\left|t_n^{B.A}(f;x) - f(x)\right| = O\left(\sum_{m=0}^{n}\frac{b_{n,m}}{m+1}(n+1)^{\beta+1}\omega(1/(n+1))\right), \text{ in view of Note 1.}$$

This completes the proof of Theorem 2.

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